Calculus is essential for the computations required to land an astronaut on the Moon. In the last chapter we introduced the definite integral as the limit of Riemann sums in the context of finding areas. However, Riemann sums and definite integrals have applications that extend far beyond the area problem. In this chapter we will show how Riemann sums and definite integrals arise in such problems as finding the volume and surface area of a solid, finding the length of a plane curve, calculating the work done by a force, finding the center of gravity of a planar region, finding the pressure and force exerted by a fluid on a submerged object, and finding properties of suspended cables.

Although these problems are diverse, the required calculations can all be approached by the same procedure that we used to find areas—breaking the required calculation into “small parts,” making an approximation for each part, adding the approximations from the parts to produce a Riemann sum that approximates the entire quantity to be calculated, and then taking the limit of the Riemann sums to produce an exact result.

6.1 AREAS BETWEEN TWO CURVES

In the last chapter we showed how to find the area between a curve \( y = f(x) \) and an interval on the \( x \)-axis. Here we will show how to find the area between two curves.

A REVIEW OF RIEMANN SUMS

Before we consider the problem of finding the area between two curves it will be helpful to review the basic principle that underlies the calculation of area as a definite integral. Recall that if \( f \) is continuous and nonnegative on \([a, b]\), then the definite integral for the area \( A \) under \( y = f(x) \) over the interval \([a, b]\) is obtained in four steps (Figure 6.1.1):

- Divide the interval \([a, b]\) into \( n \) subintervals, and use those subintervals to divide the region under the curve \( y = f(x) \) into \( n \) strips.
- Assuming that the width of the \( k \)th strip is \( \Delta x_k \), approximate the area of that strip by the area \( f(x_k^*)\Delta x_k \) of a rectangle of width \( \Delta x_k \) and height \( f(x_k^*) \), where \( x_k^* \) is a point in the \( k \)th subinterval.
- Add the approximate areas of the strips to approximate the entire area \( A \) by the Riemann sum:
  \[
  A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k
  \]
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Take the limit of the Riemann sums as the number of subintervals increases and all their widths approach zero. This causes the error in the approximations to approach zero and produces the following definite integral for the exact area

\[ A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k = \int_{a}^{b} f(x) \, dx \]

Figure 6.1.2 illustrates the effect that the limit process has on the various parts of the Riemann sum:

- The quantity \( x_k^* \) in the Riemann sum becomes the variable \( x \) in the definite integral.
- The interval width \( \Delta x_k \) in the Riemann sum becomes the \( dx \) in the definite integral.
- The interval \( [a, b] \), which is the union of the subintervals with widths \( \Delta x_1, \Delta x_2, \ldots, \Delta x_n \), does not appear explicitly in the Riemann sum but is represented by the upper and lower limits of integration in the definite integral.

**AREA BETWEEN \( y = f(x) \) AND \( y = g(x) \)**

We will now consider the following extension of the area problem.

6.1.1 **FIRST AREA PROBLEM** Suppose that \( f \) and \( g \) are continuous functions on an interval \([a, b]\) and \( f(x) \geq g(x) \) for \( a \leq x \leq b \)  

[This means that the curve \( y = f(x) \) lies above the curve \( y = g(x) \) and that the two can touch but not cross.] Find the area \( A \) of the region bounded above by \( y = f(x) \), below by \( y = g(x) \), and on the sides by the lines \( x = a \) and \( x = b \) (Figure 6.1.3a).

\[ A \approx \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)] \Delta x_k \]

To solve this problem we divide the interval \([a, b]\) into \( n \) subintervals, which has the effect of subdividing the region into \( n \) strips (Figure 6.1.3b). If we assume that the width of the \( k \)th strip is \( \Delta x_k \), then the area of the strip can be approximated by the area of a rectangle of width \( \Delta x_k \) and height \( f(x_k^*) - g(x_k^*) \), where \( x_k^* \) is a point in the \( k \)th subinterval. Adding these approximations yields the following Riemann sum that approximates the area \( A \):

\[ A \approx \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)] \Delta x_k \]

Taking the limit as \( n \) increases and the widths of all the subintervals approach zero yields the following definite integral for the area \( A \) between the curves:

\[ A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)] \Delta x_k = \int_{a}^{b} [f(x) - g(x)] \, dx \]
In summary, we have the following result.

6.1.2 AREA FORMULA If \( f \) and \( g \) are continuous functions on the interval \([a, b]\), and if \( f(x) \geq g(x) \) for all \( x \) in \([a, b]\), then the area of the region bounded above by \( y = f(x) \), below by \( y = g(x) \), on the left by the line \( x = a \), and on the right by the line \( x = b \) is

\[
A = \int_a^b [f(x) - g(x)] \, dx \tag{1}
\]

Example 1 Find the area of the region bounded above by \( y = x + 6 \), bounded below by \( y = x^2 \), and bounded on the sides by the lines \( x = 0 \) and \( x = 2 \).

Solution. The region and a cross section are shown in Figure 6.1.4. The cross section extends from \( g(x) = x^2 \) on the bottom to \( f(x) = x + 6 \) on the top. If the cross section is moved through the region, then its leftmost position will be \( x = 0 \) and its rightmost position will be \( x = 2 \). Thus, from (1)

\[
A = \int_0^2 [(x + 6) - x^2] \, dx = \left[ \frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3} \]

It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 6.1.5). When that occurs you will have to determine the points of intersection to obtain the limits of integration.

Example 2 Find the area of the region that is enclosed between the curves \( y = x^2 \) and \( y = x + 6 \).

Solution. A sketch of the region (Figure 6.1.6) shows that the lower boundary is \( y = x^2 \) and the upper boundary is \( y = x + 6 \). At the endpoints of the region, the upper and lower boundaries have the same \( y \)-coordinates; thus, to find the endpoints we equate

\[
y = x^2 \quad \text{and} \quad y = x + 6 \tag{2}
\]

This yields

\[
x^2 = x + 6 \quad \text{or} \quad x^2 - x - 6 = 0 \quad \text{or} \quad (x + 2)(x - 3) = 0
\]

from which we obtain

\[
x = -2 \quad \text{and} \quad x = 3
\]

Although the \( y \)-coordinates of the endpoints are not essential to our solution, they may be obtained from (2) by substituting \( x = -2 \) and \( x = 3 \) in either equation. This yields \( y = 4 \) and \( y = 9 \), so the upper and lower boundaries intersect at \((-2, 4)\) and \((3, 9)\).
From (1) with \( f(x) = x + 6 \), \( g(x) = x^2 \), \( a = -2 \), and \( b = 3 \), we obtain the area
\[
A = \int_{-2}^{3} [(x + 6) - x^2] \, dx = \left[ \frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^{3} = \frac{27}{2} - \left( -\frac{22}{3} \right) = \frac{125}{6}
\]

In the case where \( f \) and \( g \) are nonnegative on the interval \([a, b] \), the formula
\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx
\]
states that the area \( A \) between the curves can be obtained by subtracting the area under \( y = g(x) \) from the area under \( y = f(x) \) (Figure 6.1.7).

Example 3 Figure 6.1.8 shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same time. Give a physical interpretation of the area \( A \) between the curves over the interval \( 0 \leq t \leq T \).

Solution. From (1)
\[
A = \int_{0}^{T} [v_2(t) - v_1(t)] \, dt = \int_{0}^{T} v_2(t) \, dt - \int_{0}^{T} v_1(t) \, dt
\]
Since \( v_1 \) and \( v_2 \) are nonnegative functions on \([0, T] \), it follows from Formula (4) of Section 6.7 that the integral of \( v_1 \) over \([0, T] \) is the distance traveled by car 1 during the time interval \( 0 \leq t \leq T \), and the integral of \( v_2 \) over \([0, T] \) is the distance traveled by car 2 during the same time interval. Since \( v_1(t) \leq v_2(t) \) on \([0, T] \), car 2 travels farther than car 1 does over the time interval \( 0 \leq t \leq T \), and the area \( A \) represents the distance by which car 2 is ahead of car 1 at time \( T \).

Some regions may require careful thought to determine the integrand and limits of integration in (1). Here is a systematic procedure that you can follow to set up this formula.

Finding the Limits of Integration for the Area Between Two Curves

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point \( x \) on the \( x \)-axis, connecting the top and bottom boundaries (Figure 6.1.9a).

Step 2. The \( y \)-coordinate of the top endpoint of the line segment sketched in Step 1 will be \( f(x) \), the bottom one \( g(x) \), and the length of the line segment will be \( f(x) - g(x) \). This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is \( x = a \) and the rightmost is \( x = b \) (Figures 6.1.9b and 6.1.9c).
6.1 Area Between Two Curves

There is a useful way of thinking about this procedure:

*If you view the vertical line segment as the “cross section” of the region at the point \( x \), then Formula (1) states that the area between the curves is obtained by integrating the length of the cross section over the interval \([a, b]\).*

It is possible for the upper or lower boundary of a region to consist of two or more different curves, in which case it will be convenient to subdivide the region into smaller pieces in order to apply Formula (1). This is illustrated in the next example.

**Example 4** Find the area of the region enclosed by \( x = y^2 \) and \( y = x - 2 \).

**Solution.** To determine the appropriate boundaries of the region, we need to know where the curves \( x = y^2 \) and \( y = x - 2 \) intersect. In Example 2 we found intersections by equating the expressions for \( y \). Here it is easier to rewrite the latter equation as \( x = y + 2 \) and equate the expressions for \( x \), namely,

\[
x = y^2 \quad \text{and} \quad x = y + 2
\]

This yields

\[
y^2 = y + 2 \quad \text{or} \quad y^2 - y - 2 = 0 \quad \text{or} \quad (y + 1)(y - 2) = 0
\]

from which we obtain \( y = -1, y = 2 \). Substituting these values in either equation in (3) we see that the corresponding \( x \)-values are \( x = 1 \) and \( x = 4 \), respectively, so the points of intersection are \((1, -1)\) and \((4, 2)\) (Figure 6.1.10a).

To apply Formula (1), the equations of the boundaries must be written so that \( y \) is expressed explicitly as a function of \( x \). The upper boundary can be written as \( y = \sqrt{x} \) (rewrite \( x = y^2 \) as \( y = \pm \sqrt{x} \) and choose the + for the upper portion of the curve). The lower boundary consists of two parts:

\[
y = -\sqrt{x} \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{and} \quad y = x - 2 \quad \text{for} \quad 1 \leq x \leq 4
\]

(Figure 6.1.10b). Because of this change in the formula for the lower boundary, it is necessary to divide the region into two parts and find the area of each part separately.

From (1) with \( f(x) = \sqrt{x}, g(x) = -\sqrt{x}, a = 0, \) and \( b = 1 \), we obtain

\[
A_1 = \int_0^1 [\sqrt{x} - (-\sqrt{x})] \, dx = 2 \int_0^1 \sqrt{x} \, dx = 2 \left[ \frac{2}{3} x^{3/2} \right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}
\]

From (1) with \( f(x) = \sqrt{x}, g(x) = x - 2, a = 1, \) and \( b = 4 \), we obtain

\[
A_2 = \int_1^4 [\sqrt{x} - (x - 2)] \, dx = \int_1^4 (\sqrt{x} - x + 2) \, dx
\]

\[
= \left[ \frac{2}{3} x^{3/2} - \frac{1}{2} x^2 + 2x \right]_1^4 = \left( \frac{16}{3} - 8 + 8 \right) - \left( \frac{2}{3} - \frac{1}{2} + 2 \right) = \frac{19}{6}
\]
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Thus, the area of the entire region is

\[ A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2} \]

REVERSING THE ROLES OF x AND y

Sometimes it is much easier to find the area of a region by integrating with respect to \( y \) rather than \( x \). We will now show how this can be done.

6.1.3 SECOND AREA PROBLEM

Suppose that \( w \) and \( v \) are continuous functions of \( y \) on an interval \([c,d]\) and that

\[ w(y) \geq v(y) \text{ for } c \leq y \leq d \]

[This means that the curve \( x = w(y) \) lies to the right of the curve \( x = v(y) \) and that the two can touch but not cross.] Find the area \( A \) of the region bounded on the left by \( x = v(y) \), on the right by \( x = w(y) \), and above and below by the lines \( y = d \) and \( y = c \) (Figure 6.1.11).

Proceeding as in the derivation of (1), but with the roles of \( x \) and \( y \) reversed, leads to the following analog of 6.1.2.

6.1.4 AREA FORMULA

If \( w \) and \( v \) are continuous functions and if \( w(y) \geq v(y) \) for all \( y \) in \([c,d]\), then the area of the region bounded on the left by \( x = v(y) \), on the right by \( x = w(y) \), below by \( y = c \), and above by \( y = d \) is

\[ A = \int_c^d [w(y) - v(y)] \, dy \]  

(4)

The guiding principle in applying this formula is the same as with (1): The integrand in (4) can be viewed as the length of the horizontal cross section at an arbitrary point \( y \) on the \( y \)-axis, in which case Formula (4) states that the area can be obtained by integrating the length of the horizontal cross section over the interval \([c,d]\) on the \( y \)-axis (Figure 6.1.12).

In Example 4, we split the region into two parts to facilitate integrating with respect to \( x \). In the next example we will see that splitting this region can be avoided if we integrate with respect to \( y \).

Example 5

Find the area of the region enclosed by \( x = y^2 \) and \( y = x - 2 \), integrating with respect to \( y \).

Solution. As indicated in Figure 6.1.10 the left boundary is \( x = y^2 \), the right boundary is \( y = x - 2 \), and the region extends over the interval \(-1 \leq y \leq 2 \). However, to apply (4) the equations for the boundaries must be written so that \( x \) is expressed explicitly as a function of \( y \). Thus, we rewrite \( y = x - 2 \) as \( x = y + 2 \). It now follows from (4) that

\[ A = \int_{-1}^{2} [(y + 2) - y^2] \, dy = \left[ \frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{2} = \frac{9}{2} \]

which agrees with the result obtained in Example 4.
6.1 Area Between Two Curves 419

QUICK CHECK EXERCISES 6.1  (See page 421 for answers.)

1. An integral expression for the area of the region between the curves \( y = 20 - 3x^2 \) and \( y = e^x \) and bounded on the sides by \( x = 0 \) and \( x = 2 \) is _______.
2. An integral expression for the area of the parallelogram bounded by \( y = 2x + 8 \), \( y = 2x - 3 \), \( x = -1 \), and \( x = 5 \) is _______. The value of this integral is _______.
3. (a) The points of intersection for the circle \( x^2 + y^2 = 4 \) and the line \( y = x + 2 \) are _______ and _______.
4. The area of the region enclosed by the curves \( y = x^2 \) and \( y = \sqrt{x} \) is _______.

EXERCISE SET 6.1  □ Graphing Utility □ CAS

1–4 Find the area of the shaded region.

1. \[ y = x^2 + 1 \]
2. \[ y = \sqrt{x} \]
3. \[ y = \frac{1}{x} \]
4. \[ y = 2 - y^2 \]

5–6 Find the area of the shaded region by (a) integrating with respect to \( x \) and (b) integrating with respect to \( y \).

5. \[ y = 2x, \quad y = x^2 \]
6. \[ y^2 = 4x, \quad y = 2x - 4 \]

7–18 Sketch the region enclosed by the curves and find its area.

7. \( y = x^2, \quad y = \sqrt{x}, \quad x = \frac{1}{2}, \quad x = 1 \)
8. \( y = x^3 - 4x, \quad y = 0, \quad x = 0, \quad x = 2 \)
9. \( y = \cos 2x, \quad y = 0, \quad x = \pi/4, \quad x = \pi/2 \)
10. \( y = \sec^2 x, \quad y = 2, \quad x = -\pi/4, \quad x = \pi/4 \)
11. \( x = \sin y, \quad x = 0, \quad y = \pi/4, \quad y = 3\pi/4 \)
12. \( x^2 = y, \quad x = y - 2 \)
13. \( y = e^x, \quad y = e^{2x}, \quad x = 0, \quad x = \ln 2 \)
14. \( x = 1/y, \quad x = 0, \quad y = 1, \quad y = e \)
15. \( y = \frac{2}{1 + x^2}, \quad y = |x| \)
16. \( y = \frac{1}{\sqrt{1 - x^2}}, \quad y = 2 \)
17. \( y = 2 + |x - 1|, \quad y = -\frac{1}{x} + 7 \)
18. \( y = x, \quad y = 4x, \quad y = -x + 2 \)

19–26 Use a graphing utility, where helpful, to find the area of the region enclosed by the curves.

19. \( y = x^3 - 4x^2 + 3x, \quad y = 0 \)
20. \( y = x^3 - 2x^2, \quad y = 2x^2 - 3x \)
21. \( y = \sin x, \quad y = \cos x, \quad x = 0, \quad x = 2\pi \)
22. \( y = x^3 - 4x, \quad y = 0 \)
23. \( x = y^3 - y, \quad x = 0 \)
24. \( x = y^3 - 4y^2 + 3y, \quad y = y^2 - y \)
25. \( y = xe^{2x}, \quad y = 2|x| \)
26. \( y = \frac{1}{x\sqrt{1 - (\ln x)^2}}, \quad y = \frac{3}{x} \)

27–30 True–False Determine whether the statement is true or false. Explain your answer. [In each exercise, assume that \( f \) and \( g \) are distinct continuous functions on \([a, b]\) and that \( A \) denotes the area of the region bounded by the graphs of \( y = f(x), \quad y = g(x), \quad x = a, \) and \( x = b\).]

27. If \( f \) and \( g \) differ by a positive constant \( c \), then \( A = c(b - a) \).
28. If \( \int_a^b [f(x) - g(x)] \, dx = 3 \) then \( A = 3 \).
29. If \( \int_a^b [f(x) - g(x)] \, dx = 0 \) then the graphs of \( y = f(x) \) and \( y = g(x) \) cross at least once on \([a, b]\).
30. If \( A = \left| \int_a^b [f(x) - g(x)] \, dx \right| \) then the graphs of \( y = f(x) \) and \( y = g(x) \) don’t cross on \([a, b]\).
31. Estimate the value of $k (0 < k < 1)$ so that the region enclosed by $y = 1/\sqrt{1-x^2}$, $y = x$, $x = 0$, and $x = k$ has an area of 1 square unit.

32. Estimate the area of the region in the first quadrant enclosed by $y = \sin 2x$ and $y = \sin^{-1} x$.

33. Use a CAS to find the area enclosed by $y = 3 - 2x$ and $y = x^6 + 2x^5 - 3x^4 + x^2$.

34. Use a CAS to find the exact area enclosed by the curves $y = x^5 - 2x^3 - 3x$ and $y = x^3$.

35. Find a horizontal line $y = k$ that divides the area between $y = x^2$ and $y = 9$ into two equal parts.

36. Find a vertical line $x = k$ that divides the area enclosed by $x = \sqrt{3}$, $x = 2$, and $y = 0$ into two equal parts.

37. (a) Find the area of the region enclosed by the parabola $y = 2x - x^2$ and the x-axis.
(b) Find the value of $m$ so that the line $y = mx$ divides the region in part (a) into two regions of equal area.

38. Find the area between the curve $y = \sin x$ and the line segment joining the points $(0, 0)$ and $(\pi/6, 1/2)$ on the curve.

39–42 Use Newton’s Method (Section 4.7), where needed, to approximate the x-coordinates of the intersections of the curves to at least four decimal places, and then use those approximations to approximate the area of the region.

39. The region that lies below the curve $y = \sin x$ and above the line $y = 0.2x$, where $x \geq 0$.

40. The region enclosed by the graphs of $y = x^2$ and $y = \cos x$.

41. The region enclosed by the graphs of $y = (\ln x)/x$ and $y = x - 2$.

42. The region enclosed by the graphs of $y = 3 - 2\cos x$ and $y = 2/(1 + x^2)$.

43. Find the area of the region that is enclosed by the curves $y = x^2 - 1$ and $y = 2\sin x$.

44. Referring to the accompanying figure, use a CAS to estimate the value of $k$ so that the areas of the shaded regions are equal.

31. The accompanying figure shows acceleration versus time curves for two cars that move along a straight track, accelerating from rest at the starting line. What does the area $A$ between the curves over the interval $0 \leq t \leq T$ represent? Justify your answer.

47. The curves in the accompanying figure model the birth rates and death rates (in millions of people per year) for a country over a 50-year period. What does the area $A$ between the curves over the interval [1960, 2010] represent? Justify your answer.

48. The accompanying figure shows the rate at which transdermal medication is absorbed into the bloodstream of an individual, as well as the rate at which the medication is eliminated from the bloodstream by metabolism. Both rates are in units of micrograms per hour ($\mu g/h$) and are displayed over an 8-hour period. What does the area $A$ between the curves over the interval [0, 8] represent? Justify your answer.

49. Find the area of the region enclosed between the curve $x^{1/2} + y^{1/2} = a^{1/2}$ and the coordinate axes.

50. Show that the area of the ellipse in the accompanying figure is $\pi ab$. [Hint: Use a formula from geometry.]
51. **Writing** Suppose that \( f \) and \( g \) are continuous on \([a, b]\) but that the graphs of \( y = f(x) \) and \( y = g(x) \) cross several times. Describe a step-by-step procedure for determining the area bounded by the graphs of \( y = f(x) \), \( y = g(x) \), \( x = a \), and \( x = b \).

**QUICK CHECK ANSWERS 6.1**

1. \( \int_{0}^{2} [(20 - 3x^2) - e^x] \, dx \)
2. \( \int_{-1}^{5} [(2x + 8) - (2x - 3)] \, dx \)
3. (a) \((-2, 0)\); \((0, 2)\) (b) \( \int_{-2}^{0} [\sqrt{4 - x^2} - (x + 2)] \, dx \)
4. \( \frac{5}{12} \)

6.2 **VOLUMES BY SLICING; DISKS AND WASHERS**

In the last section we showed that the area of a plane region bounded by two curves can be obtained by integrating the length of a general cross section over an appropriate interval. In this section we will see that the same basic principle can be used to find volumes of certain three-dimensional solids.

**VOLUMES BY SLICING**

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the area. Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 6.2.1).

What makes this method work is the fact that a thin slab has a cross section that does not vary much in size or shape, which, as we will see, makes its volume easy to approximate (Figure 6.2.2). Moreover, the thinner the slab, the less variation in its cross sections and the better the approximation. Thus, once we approximate the volumes of the slabs, we can set up a Riemann sum whose limit is the volume of the entire solid. We will give the details shortly, but first we need to discuss how to find the volume of a solid whose cross sections do not vary in size and shape (i.e., are congruent).

One of the simplest examples of a solid with congruent cross sections is a right circular cylinder of radius \( r \), since all cross sections taken perpendicular to the central axis are circular regions of radius \( r \). The volume \( V \) of a right circular cylinder of radius \( r \) and height \( h \) can be expressed in terms of the height and the area of a cross section as

\[
V = \pi r^2 h = [\text{area of a cross section}] \times [\text{height}]
\]
This is a special case of a more general volume formula that applies to solids called right cylinders. A right cylinder is a solid that is generated when a plane region is translated along a line or axis that is perpendicular to the region (Figure 6.2.3).

If a right cylinder is generated by translating a region of area \( A \) through a distance \( h \), then \( h \) is called the **height** (or sometimes the **width**) of the cylinder, and the volume \( V \) of the cylinder is defined to be

\[
V = A \cdot h = [\text{area of a cross section}] \times [\text{height}]
\]

(Figure 6.2.4). Note that this is consistent with Formula (1) for the volume of a right circular cylinder.

We now have all of the tools required to solve the following problem.

**6.2.1 Problem** Let \( S \) be a solid that extends along the \( x \)-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the \( x \)-axis at \( x = a \) and \( x = b \) (Figure 6.2.5). Find the volume \( V \) of the solid, assuming that its cross-sectional area \( A(x) \) is known at each \( x \) in the interval \([a, b]\).

To solve this problem we begin by dividing the interval \([a, b]\) into \( n \) subintervals, thereby dividing the solid into \( n \) slabs as shown in the left part of Figure 6.2.6. If we assume that the width of the \( k \)th subinterval is \( \Delta x_k \), then the volume of the \( k \)th slab can be approximated by the volume \( A(x_k^*) \Delta x_k \) of a right cylinder of width (height) \( \Delta x_k \) and cross-sectional area \( A(x_k^*) \), where \( x_k^* \) is a point in the \( k \)th subinterval (see the right part of Figure 6.2.6).

Adding these approximations yields the following Riemann sum that approximates the volume \( V \):

\[
V \approx \sum_{k=1}^{n} A(x_k^*) \Delta x_k
\]
Taking the limit as \( n \) increases and the widths of all the subintervals approach zero yields the definite integral

\[
V = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*) \Delta x_k = \int_{a}^{b} A(x) \, dx
\]

In summary, we have the following result.

**6.2.2 VOLUME FORMULA** Let \( S \) be a solid bounded by two parallel planes perpendicular to the \( x \)-axis at \( x = a \) and \( x = b \). If, for each \( x \) in \([a, b]\), the cross-sectional area of \( S \) perpendicular to the \( x \)-axis is \( A(x) \), then the volume of the solid is

\[
V = \int_{a}^{b} A(x) \, dx
\]

provided \( A(x) \) is integrable.

There is a similar result for cross sections perpendicular to the \( y \)-axis.

**6.2.3 VOLUME FORMULA** Let \( S \) be a solid bounded by two parallel planes perpendicular to the \( y \)-axis at \( y = c \) and \( y = d \). If, for each \( y \) in \([c, d]\), the cross-sectional area of \( S \) perpendicular to the \( y \)-axis is \( A(y) \), then the volume of the solid is

\[
V = \int_{c}^{d} A(y) \, dy
\]

provided \( A(y) \) is integrable.

In words, these formulas state:

**The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.**

**Example 1** Derive the formula for the volume of a right pyramid whose altitude is \( h \) and whose base is a square with sides of length \( a \).

**Solution.** As illustrated in Figure 6.2.7a, we introduce a rectangular coordinate system in which the \( y \)-axis passes through the apex and is perpendicular to the base, and the \( x \)-axis passes through the base and is parallel to a side of the base.

At any \( y \) in the interval \([0, h]\) on the \( y \)-axis, the cross section perpendicular to the \( y \)-axis is a square. If \( s \) denotes the length of a side of this square, then by similar triangles (Figure 6.2.7b)

\[
\frac{\frac{1}{2} s}{\frac{1}{2} a} = \frac{h-y}{h} \quad \text{or} \quad s = \frac{a}{h} (h-y)
\]

Thus, the area \( A(y) \) of the cross section at \( y \) is

\[
A(y) = s^2 = \frac{a^2}{h^2} (h-y)^2
\]
and by (4) the volume is

\[ V = \int_0^h A(y) \, dy = \int_0^h \frac{a^2}{h^2} (h - y)^2 \, dy = \frac{a^2}{h^2} \int_0^h (h - y)^2 \, dy \]

\[ = \frac{a^2}{h^2} \left[ -\frac{1}{3} (h - y)^3 \right]_y=0 \]

\[ = \frac{a^2}{h^2} \left[ 0 + \frac{1}{3} h^3 \right] = \frac{1}{3} a^2 h \]

That is, the volume is \( \frac{1}{3} \) of the area of the base times the altitude.

- **SOLIDS OF REVOLUTION**

  A solid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the axis of revolution. Many familiar solids are of this type (Figure 6.2.8).

- **VOLUMES BY DISKS PERPENDICULAR TO THE x-AXIS**

  We will be interested in the following general problem.

  **6.2.4 PROBLEM** Let \( f \) be continuous and nonnegative on \([a, b]\), and let \( R \) be the region that is bounded above by \( y = f(x) \), below by the \( x \)-axis, and on the sides by the lines \( x = a \) and \( x = b \) (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region \( R \) about the \( x \)-axis.
6.2 Volumes by Slicing: Disks and Washers 425

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is a circular disk of radius \( f(x) \) (Figure 6.2.9b). The area of this region is

\[
A(x) = \pi [f(x)]^2
\]

Thus, from (3) the volume of the solid is

\[
V = \int_a^b \pi [f(x)]^2 \, dx
\]

Because the cross sections are disk shaped, the application of this formula is called the method of disks.

Example 2 Find the volume of the solid that is obtained when the region under the curve \( y = \sqrt{x} \) over the interval \([1, 4]\) is revolved about the x-axis (Figure 6.2.10).

Solution. From (5), the volume is

\[
V = \int_1^4 \pi x \, dx = \pi \left[ \frac{x^2}{2} \right]_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}
\]

Example 3 Derive the formula for the volume of a sphere of radius \( r \).

Solution. As indicated in Figure 6.2.11, a sphere of radius \( r \) can be generated by revolving the upper semicircular disk enclosed between the x-axis and

\[
x^2 + y^2 = r^2
\]

about the x-axis. Since the upper half of this circle is the graph of \( y = f(x) = \sqrt{r^2 - x^2} \), it follows from (5) that the volume of the sphere is

\[
V = \int_{-r}^r \pi (r^2 - x^2) \, dx = \pi \left[ \frac{r^2 x - \frac{x^3}{3}}{3} \right]_{-r}^r = \frac{4}{3}\pi r^3
\]

VOLUMES BY WASHERS PERPENDICULAR TO THE x-AXIS

Not all solids of revolution have solid interiors; some have holes or channels that create interior surfaces, as in Figure 6.2.8d. So we will also be interested in problems of the following type.

6.2.5 Problem Let \( f \) and \( g \) be continuous and nonnegative on \([a, b]\), and suppose that \( f(x) \geq g(x) \) for all \( x \) in the interval \([a, b]\). Let \( R \) be the region that is bounded above by \( y = f(x) \), below by \( y = g(x) \), and on the sides by the lines \( x = a \) and \( x = b \) (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region \( R \) about the x-axis (Figure 6.2.12b).

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point \( x \) is the annular or “washer-shaped”
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region with inner radius \( g(x) \) and outer radius \( f(x) \) (Figure 6.2.12b); its area is

\[
A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi((f(x))^2 - (g(x))^2)
\]

Thus, from (3) the volume of the solid is

\[
V = \int_a^b \pi([f(x)]^2 - [g(x)]^2) \, dx
\]  

(6)

Because the cross sections are washer shaped, the application of this formula is called the method of washers.

<table>
<thead>
<tr>
<th>Example 4</th>
<th>Find the volume of the solid generated when the region between the graphs of the equations ( f(x) = \frac{1}{2} + x^2 ) and ( g(x) = x ) over the interval ([0, 2]) is revolved about the ( x )-axis.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution</td>
<td>First sketch the region (Figure 6.2.13a); then imagine revolving it about the ( x )-axis (Figure 6.2.13b). From (6) the volume is</td>
</tr>
</tbody>
</table>
|           | \[
V = \int_a^b \pi([f(x)]^2 - [g(x)]^2) \, dx = \int_0^2 \pi \left( \frac{1}{4} + x^4 \right) \, dx
\] |
|           | \[
= \pi \left( \frac{1}{4} + x^4 \right) \bigg|_0^2 = \pi \left( \frac{3}{5} \right) = 69\pi \frac{10}{10}
\] |
| Figure 6.2.13 | Unequal scales on axes |
| (a) | (b) |

**VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE \( y \)-AXIS**

The methods of disks and washers have analogs for regions that are revolved about the \( y \)-axis (Figures 6.2.14 and 6.2.15). Using the method of slicing and Formula (4), you should be able to deduce the following formulas for the volumes of the solids in the figures.

\[
V = \int_c^d \pi[u(y)]^2 \, dy \quad \text{Disks}
\]

\[
V = \int_c^d \pi([w(y)]^2 - [v(y)]^2) \, dy \quad \text{Washers}
\]

(7–8)
6.2 Volumes by Slicing; Disks and Washers

Example 5  Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the $y$-axis.

Solution.  First sketch the region and the solid (Figure 6.2.16). The cross sections taken perpendicular to the $y$-axis are disks, so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, from (7) with $u(y) = y^2$, the volume is

$$V = \int_{c}^{d} \pi[u(y)]^2 \, dy = \int_{0}^{2} \pi y^4 \, dy = \frac{\pi y^5}{5} \bigg|_{0}^{2} = \frac{32\pi}{5}.$$

OTHER AXES OF REVOLUTION

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (3) and (4) and integrate an appropriate cross-sectional area to find the volume.

Example 6  Find the volume of the solid generated when the region under the curve $y = x^2$ over the interval $[0, 2]$ is rotated about the line $y = -1$.

Solution.  First sketch the region and the axis of revolution; then imagine revolving the region about the axis (Figure 6.2.17). At each $x$ in the interval $0 \leq x \leq 2$, the cross section of the solid perpendicular to the axis $y = -1$ is a washer with outer radius $x^2 + 1$ and inner radius 1. Since the area of this washer is

$$A(x) = \pi((x^2 + 1)^2 - 1^2) = \pi(x^4 + 2x^2).$$
it follows by (3) that the volume of the solid is
\[ V = \int_0^2 A(x) \, dx = \int_0^2 \pi \left( x^4 + 2x^2 \right) \, dx = \pi \left[ \frac{1}{5} x^5 + \frac{2}{3} x^3 \right]_0^2 = \frac{176\pi}{15} \]

Figure 6.2.17

**QUICK CHECK EXERCISES 6.2** (See page 431 for answers.)

1. A solid \( S \) extends along the \( x \)-axis from \( x = 1 \) to \( x = 3 \). For \( x \) between 1 and 3, the cross-sectional area of \( S \) perpendicular to the \( x \)-axis is \( 3x^2 \). An integral expression for the volume of \( S \) is _______. The value of this integral is _______.

2. A solid \( S \) is generated by revolving the region between the \( x \)-axis and the curve \( y = \sqrt{\sin x} \) (\( 0 \leq x \leq \pi \)) about the \( x \)-axis.
   (a) For \( x \) between 0 and \( \pi \), the cross-sectional area of \( S \) perpendicular to the \( x \)-axis at \( x \) is \( A(x) = _____ \).
   (b) An integral expression for the volume of \( S \) is _______.
   (c) The value of the integral in part (b) is _______.

3. A solid \( S \) is generated by revolving the region enclosed by the line \( y = 2x + 1 \) and the curve \( y = x^2 + 1 \) about the \( x \)-axis.
   (a) For \( x \) between ______ and _______, the cross-sectional area of \( S \) perpendicular to the \( x \)-axis at \( x \) is \( A(x) = _____ \).
   (b) An integral expression for the volume of \( S \) is _______.

4. A solid \( S \) is generated by revolving the region enclosed by the line \( y = x + 1 \) and the curve \( y = x^2 + 1 \) about the \( y \)-axis.
   (a) For \( y \) between ______ and _______, the cross-sectional area of \( S \) perpendicular to the \( y \)-axis at \( y \) is \( A(y) = _____ \).
   (b) An integral expression for the volume of \( S \) is _______.

**EXERCISE SET 6.2**

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis.

1. \( y = \sqrt{3 - x} \)

2. \( y = x \)

3. \( y = 2 - x^2 \)

4. \( y = 1/x \)
Find the volume of the solid that results when the region enclosed by the given curves is revolved about the $x$-axis.

9. $y = \sqrt{25-x^2}$, $y = 3$

10. $y = 9 - x^2$, $y = 0$

11. $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$

12. $y = e^x$, $y = 0$, $x = 0$, $x = 1$

13. $y = 1/x^2$, $x = -2$, $x = 2$, $y = 0$

14. $y = e^{-2x}$, $y = 0$, $x = 0$, $x = 1$

15. $y = e^{3x}$, $x = 0$, $x = 1$, $y = 0$

Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the $x$-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the $x$-axis are squares.

16. $y = x^2$, $y = 0$

17. $y = x^3$, $y = 1$

Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the $x$-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the $x$-axis are squares.

18. $y = 1$, $x = 0$

Find the volume of the solid whose base is the region enclosed by the given curves is revolved about the $x$-axis.

19. $y = \sqrt{1-x^2}$, $y = 0$

20. $y = \sqrt{\sqrt{2} - x^2}$, $y = 0$

21. $y = \sqrt{x}$, $y = 1$

22. $y = 1 - x^2$, $y = 0$

23. $x = y^2$, $x = y + 2$

24. $x = 1 - y^2$, $x = 2 + y^2$, $y = -1$, $y = 1$

25. $y = \ln x$, $x = 0$, $y = 0$, $y = 1$

26. $y = \sqrt{1-x^2}$, $(x > 0)$, $x = 0$, $y = 0$, $y = 2$

27. If each cross section of $S$ perpendicular to the $x$-axis is a square, then $S$ is a rectangular parallelepiped (i.e., is box shaped).

28. If each cross section of $S$ is a disk or a washer, then $S$ is a solid of revolution.

29. If $x$ is in centimeters (cm), then $A(x)$ must be a quadratic function of $x$, since units of $A(x)$ will be square centimeters (cm$^2$).

30. The average value of $A(x)$ on the interval $[a, b]$ is given by $V/(b-a)$.

31. Find the volume of the solid that results when the region above the $x$-axis and below the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

is revolved about the $x$-axis.

32. Let $V$ be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ ($0 < b < 2$) is revolved about the $x$-axis. Find the value of $b$ for which $V = 3$.

33. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x+1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the $x$-axis. [Hint: Split the solid into two parts.]

34. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 6-x$, and $y = 0$ is revolved about the $x$-axis. [Hint: Split the solid into two parts.]

35. Suppose that $f$ is a continuous function on $[a, b]$, and let $R$ be the region between the curve $y = f(x)$ and the line $y = k$ from $x = a$ to $x = b$. Using the method of disks, derive with explanation a formula for the volume of a solid generated by revolving $R$ about the line $y = k$. State and explain additional assumptions, if any, that you need about $f$ for your formula.

36. Suppose that $u$ and $w$ are continuous functions on $[c, d]$, and let $R$ be the region between the curves $x = u(y)$ and $x = w(y)$ from $y = c$ to $y = d$. Using the method of washers, derive with explanation a formula for the volume of a solid generated by revolving $R$ about the line

6.2 Volumes by Slicing; Disks and Washers

FOCUS ON CONCEPTS
39. Find the volume of the solid that results when the region enclosed by \( y = \sqrt{x}, \ y = 0, \) and \( x = 9 \) is revolved about the line \( x = 9. \)

40. Find the volume of the solid that results when the region in Exercise 39 is revolved about the line \( y = 3. \)

41. Find the volume of the solid that results when the region enclosed by \( x = y^2 \) and \( x = y \) is revolved about the line \( y = -1. \)

42. Find the volume of the solid that results when the region in Exercise 41 is revolved about the line \( x = -1. \)

43. Find the volume of the solid that results when the region enclosed by \( y = x^2 \) and \( y = x^3 \) is revolved about the line \( x = 1. \)

44. Find the volume of the solid that results when the region in Exercise 43 is revolved about the line \( y = -1. \)

45. A nose cone for a space reentry vehicle is designed so that a cross section, taken \( x \) ft from the tip and perpendicular to the axis of symmetry, is a circle of radius \( \frac{1}{4} x^2 \) ft. Find the volume of the nose cone given that its length is 20 ft.

46. A certain solid is 1 ft high, and a horizontal cross section taken \( x \) ft above the bottom of the solid is an annulus of inner radius \( x^2 \) ft and outer radius \( \sqrt{x} \) ft. Find the volume of the solid.

47. Find the volume of the solid whose base is the region bounded between the curves \( y = x \) and \( y = x^2, \) and whose cross sections perpendicular to the \( x \)-axis are squares.

48. The base of a certain solid is the region enclosed by \( y = \sqrt{x}, \ y = 0, \) and \( x = 4. \) Every cross section perpendicular to the \( x \)-axis is a semicircle with its diameter across the base. Find the volume of the solid.

49. In parts (a)–(c) find the volume of the solid whose base is enclosed by the circle \( x^2 + y^2 = 1 \) and whose cross sections taken perpendicular to the \( x \)-axis are

   (a) semicircles  
   (b) squares  
   (c) equilateral triangles.

   (a)  
   (b)  
   (c)

50. As shown in the accompanying figure, a cathedral dome is designed with three semicircular supports of radius \( r \) so that each horizontal cross section is a regular hexagon. Show that the volume of the dome is \( r^3 \sqrt{3}. \)

51–54 Use a CAS to estimate the volume of the solid that results when the region enclosed by the curves is revolved about the stated axis.

51. \( y = \sin^5 x, \ y = 2x/\pi, \ x = 0, \ x = \pi/2; \ x \)-axis

52. \( y = \pi^2 \sin x \cos^3 x, \ y = 4x^2, \ x = 0, \ x = \pi/4; \ x \)-axis

53. \( y = e^x, \ x = 1, \ y = 1; \ y \)-axis

54. \( y = x \sqrt{\tan^{-1} x}, \ y = x; \ x \)-axis

55. The accompanying figure shows a spherical cap of radius \( r \) and height \( h \) cut from a sphere of radius \( r. \) Show that the volume \( V \) of the spherical cap can be expressed as

   (a) \( V = \frac{1}{2} \pi r^2 (3r - h) \)  
   (b) \( V = \frac{1}{6} \pi h (3r^2 + h^2). \)

56. If fluid enters a hemispherical bowl with a radius of 10 ft at a rate of \( \frac{1}{2} \) ft³/min, how fast will the fluid be rising when the depth is 5 ft? [Hint: See Exercise 55.]

57. The accompanying figure (on the next page) shows the dimensions of a small lightbulb at 10 equally spaced points.

(a) Use formulas from geometry to make a rough estimate of the volume enclosed by the glass portion of the bulb.
6.2 Volumes by Slicing; Disks and Washers

(b) Use the average of left and right endpoint approximations to approximate the volume.

\[
\begin{array}{c}
\text{Figure Ex-57} \\
\end{array}
\]

58. Use the result in Exercise 55 to find the volume of the solid that remains when a hole of radius \(r/2\) is drilled through the center of a sphere of radius \(r\), and then check your answer by integrating.

59. As shown in the accompanying figure, a cocktail glass with a bowl shaped like a hemisphere of diameter 8 cm contains a cherry with a diameter of 2 cm. If the glass is filled to a depth of \(h\) cm, what is the volume of liquid it contains? [Hint: First consider the case where the cherry is partially submerged, then the case where it is totally submerged.]

\[
\begin{array}{c}
\text{Figure Ex-59} \\
\end{array}
\]

60. Find the volume of the torus that results when the region enclosed by the circle of radius \(r\) with center at \((h, 0), h > r,\) is revolved about the \(y\)-axis. [Hint: Use an appropriate formula from plane geometry to help evaluate the definite integral.]

61. A wedge is cut from a right circular cylinder of radius \(r\) by two planes, one perpendicular to the axis of the cylinder and the other making an angle \(\theta\) with the first. Find the volume of the wedge by slicing perpendicular to the \(y\)-axis as shown in the accompanying figure.

\[
\begin{array}{c}
\text{Figure Ex-61} \\
\end{array}
\]

62. Find the volume of the wedge described in Exercise 61 by slicing perpendicular to the \(x\)-axis.

63. Two right circular cylinders of radius \(r\) have axes that intersect at right angles. Find the volume of the solid common to the two cylinders. [Hint: One-eighth of the solid is sketched in the accompanying figure.]

64. In 1635 Bonaventura Cavalieri, a student of Galileo, stated the following result, called \textbf{Cavalieri’s principle}: If two solids have the same height, and if the areas of their cross sections taken parallel to and at equal distances from their bases are always equal, then the solids have the same volume. Use this result to find the volume of the oblique cylinder in the accompanying figure. (See Exercise 52 of Section 6.1 for a planar version of Cavalieri’s principle.)

\[
\begin{array}{c}
\text{Figure Ex-63} \\
\end{array}
\]

65. \textbf{Writing} Use the results of this section to derive Cavalieri’s principle (Exercise 64).

66. \textbf{Writing} Write a short paragraph that explains how Formulas (4)–(8) may all be viewed as consequences of Formula (3).

\[
\begin{array}{c}
\text{Figure Ex-64} \\
\end{array}
\]

\textbf{Quick Check Answers 6.2}

1. \(\int_1^3 3x^2 \, dx; \) 26  
2. (a) \(\pi \sin x\) (b) \(\int_0^\pi \pi \sin x \, dx\) (c) \(2\pi\)  
3. (a) 0; 2; \(\pi[(2x + 1)^2 - (x^2 + 1)^2] = \pi[-x^4 + 2x^2 + 4x]\)  
(b) \(\int_0^2 \pi[-x^4 + 2x^2 + 4x] \, dx\)  
4. (a) 1; 2; \(\pi[(y - 1) - (y - 1)^2] = \pi[-y^2 + 3y - 2]\) (b) \(\int_1^2 \pi[-y^2 + 3y - 2] \, dy\)
The methods for computing volumes that have been discussed so far depend on our ability to compute the cross-sectional area of the solid and to integrate that area across the solid. In this section we will develop another method for finding volumes that may be applicable when the cross-sectional area cannot be found or the integration is too difficult.

CYLINDRICAL SHELLS

In this section we will be interested in the following problem.

6.3.1 Problem Let \( f \) be continuous and nonnegative on \([a, b]\) \((0 \leq a < b)\), and let \( R \) be the region that is bounded above by \( y = f(x) \), below by the \( x \)-axis, and on the sides by the lines \( x = a \) and \( x = b \). Find the volume \( V \) of the solid of revolution \( S \) that is generated by revolving the region \( R \) about the \( y \)-axis (Figure 6.3.1).

Sometimes problems of the above type can be solved by the method of disks or washers perpendicular to the \( y \)-axis, but when that method is not applicable or the resulting integral is difficult, the method of cylindrical shells, which we will discuss here, will often work.

A cylindrical shell is a solid enclosed by two concentric right circular cylinders (Figure 6.3.2). The volume \( V \) of a cylindrical shell with inner radius \( r_1 \), outer radius \( r_2 \), and height \( h \) can be written as

\[
V = \pi (r_2^2 - r_1^2)h
\]

But \( \frac{1}{2}(r_1 + r_2) \) is the average radius of the shell and \( r_2 - r_1 \) is its thickness, so

\[
V = 2\pi \left[ \frac{1}{2}(r_1 + r_2) \right] \cdot h \cdot (r_2 - r_1)
\]

We will now show how this formula can be used to solve Problem 6.3.1. The underlying idea is to divide the interval \([a, b]\) into \( n \) subintervals, thereby subdividing the region \( R \) into \( n \) strips, \( R_1, R_2, \ldots, R_n \) (Figure 6.3.3a). When the region \( R \) is revolved about the \( y \)-axis, these strips generate “tube-like” solids \( S_1, S_2, \ldots, S_n \) that are nested one inside the other and together comprise the entire solid \( S \) (Figure 6.3.3b). Thus, the volume \( V \) of the solid can be obtained by adding together the volumes of the tubes; that is,

\[
V = V(S_1) + V(S_2) + \cdots + V(S_n)
\]
As a rule, the tubes will have curved upper surfaces, so there will be no simple formulas for their volumes. However, if the strips are thin, then we can approximate each strip by a rectangle (Figure 6.3.4a). These rectangles, when revolved about the \( y \)-axis, will produce cylindrical shells whose volumes closely approximate the volumes of the tubes generated by the original strips (Figure 6.3.4b). We will show that by adding the volumes of the cylindrical shells we can obtain a Riemann sum that approximates the volume \( V \), and by taking the limit of the Riemann sums we can obtain an integral for the exact volume \( V \).

To implement this idea, suppose that the \( k \)th strip extends from \( x_{k-1} \) to \( x_k \) and that the width of this strip is \( \Delta x_k = x_k - x_{k-1} \). If we let \( x_k^* \) be the midpoint of the interval \( [x_{k-1}, x_k] \), and if we construct a rectangle of height \( f(x_k^*) \) over the interval, then revolving this rectangle about the \( y \)-axis produces a cylindrical shell of average radius \( x_k^* \), height \( f(x_k^*) \), and thickness \( \Delta x_k \) (Figure 6.3.5). From (1), the volume \( V_k \) of this cylindrical shell is

\[
V_k = 2\pi x_k^* f(x_k^*) \Delta x_k
\]

Adding the volumes of the \( n \) cylindrical shells yields the following Riemann sum that approximates the volume \( V \):

\[
V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x_k
\]

Taking the limit as \( n \) increases and the widths of all the subintervals approach zero yields the definite integral

\[
V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x_k = \int_{a}^{b} 2\pi x f(x) \, dx
\]

In summary, we have the following result.
6.3.2 VOLUME BY CYLINDRICAL SHELLS ABOUT THE y-AXIS Let \( f \) be continuous and nonnegative on \([a, b]\) \((0 \leq a < b)\), and let \( R \) be the region that is bounded above by \( y = f(x) \), below by the \( x \)-axis, and on the sides by the lines \( x = a \) and \( x = b \). Then the volume \( V \) of the solid of revolution that is generated by revolving the region \( R \) about the \( y \)-axis is given by

\[
V = \int_a^b 2\pi xf(x) \, dx \tag{2}
\]

Example 1 Use cylindrical shells to find the volume of the solid generated when the region enclosed between \( y = \sqrt{x}, x = 1, x = 4 \), and the \( x \)-axis is revolved about the \( y \)-axis.

Solution. First sketch the region (Figure 6.3.6a); then imagine revolving it about the \( y \)-axis (Figure 6.3.6b). Since \( f(x) = \sqrt{x}, a = 1 \), and \( b = 4 \), Formula (2) yields

\[
V = \int_1^4 2\pi x \sqrt{x} \, dx = 2\pi \int_1^4 x^{3/2} \, dx = \left[ \frac{2\pi}{5} x^{5/2} \right]_1^4 = \frac{4\pi}{5} [32 - 1] = \frac{124\pi}{5} \]

VARIATIONS OF THE METHOD OF CYLINDRICAL SHELLS

The method of cylindrical shells is applicable in a variety of situations that do not fit the conditions required by Formula (2). For example, the region may be enclosed between two curves, or the axis of revolution may be some line other than the \( y \)-axis. However, rather than develop a separate formula for every possible situation, we will give a general way of thinking about the method of cylindrical shells that can be adapted to each new situation as it arises.

For this purpose, we will need to reexamine the integrand in Formula (2): At each \( x \) in the interval \([a, b]\), the vertical line segment from the \( x \)-axis to the curve \( y = f(x) \) can be viewed as the cross section of the region \( R \) at \( x \) (Figure 6.3.7a). When the region \( R \) is revolved about the \( y \)-axis, the cross section at \( x \) sweeps out the surface of a right circular cylinder of height \( f(x) \) and radius \( x \) (Figure 6.3.7b). The area of this surface is \( 2\pi xf(x) \) (Figure 6.3.7c), which is the integrand in (2). Thus, Formula (2) can be viewed informally in the following way.

6.3.3 AN INFORMAL VIEWPOINT ABOUT CYLINDRICAL SHELLS The volume \( V \) of a solid of revolution that is generated by revolving a region \( R \) about an axis can be obtained by integrating the area of the surface generated by an arbitrary cross section of \( R \) taken parallel to the axis of revolution.
6.3 **Volumes by Cylindrical Shells** 435

The following examples illustrate how to apply this result in situations where Formula (2) is not applicable.

**Example 2** Use cylindrical shells to find the volume of the solid generated when the region \( R \) in the first quadrant enclosed between \( y = x \) and \( y = x^2 \) is revolved about the \( y \)-axis (Figure 6.3.8a).

**Solution.** As illustrated in part (b) of Figure 6.3.8, at each \( x \) in \([0, 1]\) the cross section of \( R \) parallel to the \( y \)-axis generates a cylindrical surface of height \( x - x^2 \) and radius \( x \). Since the area of this surface is

\[
2\pi x \cdot (x - x^2)
\]

the volume of the solid is

\[
V = \int_0^1 2\pi x (x - x^2) \, dx = 2\pi \int_0^1 (x^2 - x^3) \, dx
\]

\[
= 2\pi \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 2\pi \left[ \frac{1}{3} - \frac{1}{4} \right] = \frac{\pi}{6}
\]

\[\blacktriangleleft\]

This solid looks like a bowl with a cone-shaped interior.

**Example 3** Use cylindrical shells to find the volume of the solid generated when the region \( R \) under \( y = x^2 \) over the interval \([0, 2]\) is revolved about the line \( y = -1 \).

**Solution.** First draw the axis of revolution; then imagine revolving the region about the axis (Figure 6.3.9a). As illustrated in Figure 6.3.9b, at each \( y \) in the interval \( 0 \leq y \leq 4 \), the cross section of \( R \) parallel to the \( x \)-axis generates a cylindrical surface of height \( 2 - \sqrt{y} \) and radius \( y + 1 \). Since the area of this surface is

\[
2\pi (y + 1)(2 - \sqrt{y})
\]

it follows that the volume of the solid is

\[
\int_0^4 2\pi (y + 1)(2 - \sqrt{y}) \, dy = 2\pi \int_0^4 (2y - y^{3/2} + 2 - y^{1/2}) \, dy
\]

\[
= 2\pi \left[ y^2 - \frac{2}{3} y^{5/2} + 2y - \frac{2}{3} y^{3/2} \right]_0^4 = \frac{176\pi}{15}
\]

\[\blacktriangleleft\]
Quick Check Exercises 6.3 (See page 438 for answers.)

1. Let \( R \) be the region between the \( x \)-axis and the curve \( y = 1 + \sqrt{x} \) for \( 1 \leq x \leq 4 \).
   (a) For \( x \) between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of \( R \) at \( x \) about the \( y \)-axis is \( \underline{\text{________}} \).
   (b) Using cylindrical shells, an integral expression for the volume of the solid generated by revolving \( R \) about the \( y \)-axis is \( \underline{\text{________}} \).

2. Let \( R \) be the region described in Quick Check Exercise 1.
   (a) For \( x \) between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of \( R \) at \( x \) about the line \( x = 5 \) is \( \underline{\text{________}} \).
   (b) Using cylindrical shells, an integral expression for the volume of the solid generated by revolving \( R \) about the line \( x = 5 \) is \( \underline{\text{________}} \).

3. A solid \( S \) is generated by revolving the region enclosed by the curves \( x = (y - 2)^2 \) and \( x = 4 \) about the \( x \)-axis. Using cylindrical shells, an integral expression for the volume of \( S \) is \( \underline{\text{________}} \).

Exercise Set 6.3

1–4 Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis.

5–12 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the \( y \)-axis.

5. \( y = x^3, \ x = 1, \ y = 0 \)
6. \( y = \sqrt{x}, \ x = 4, \ x = 9, \ y = 0 \)
7. \( y = 1/x, \ y = 0, \ x = 1, \ x = 3 \)
8. \( y = \cos(x^2), \ x = 0, \ x = 1/2 \sqrt{\pi}, \ y = 0 \)
9. \( y = 2x - 1, \ y = -2x + 3, \ x = 2 \)
10. \( y = 2x - x^2, \ y = 0 \)
11. \( y = \frac{1}{x^2 + 1}, \ x = 0, \ x = 1, \ y = 0 \)
12. \( y = e^{x^2}, \ x = 1, \ x = \sqrt{3}, \ y = 0 \)

13–16 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the \( x \)-axis.
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25. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve
\[ y = x^3 - 3x^2 + 2x \]
over \([0, 1]\) is revolved about the \(y\)-axis.

(b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.

26. Let \(f\) be continuous and nonnegative on \([a, b]\), and let \(R\) be the region that is enclosed by \(y = f(x)\) and \(y = 0\) for \(a \leq x \leq b\). Using the method of cylindrical shells, derive with explanation a formula for the volume of the solid generated by revolving \(R\) about the line \(x = k\), where \(k \leq a\).

27–28 Using the method of cylindrical shells, set up but do not evaluate an integral for the volume of the solid generated when the region \(R\) is revolved about (a) the line \(x = 1\) and (b) the line \(y = -1\).

27. \(R\) is the region bounded by the graphs of \(y = x\), \(y = 0\), and \(x = 1\).

28. \(R\) is the region in the first quadrant bounded by the graphs of \(y = \sqrt{1-x^2}\), \(y = 0\), and \(x = 0\).

29. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by \(y = \frac{1}{x^3}\), \(x = 1\), \(x = 2\), \(y = 0\) is revolved about the line \(x = -1\).

30. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by \(y = x^3\), \(y = 1\), \(x = 0\) is revolved about the line \(y = 1\).

31. Use cylindrical shells to find the volume of the solid of revolution of the triangle with vertices \((0, 0), (0, r), (h, 0)\), where \(r > 0\) and \(h > 0\), is revolved about the \(x\)-axis.

32. The region enclosed between the curve \(y^2 = kx\) and the line \(x = \frac{1}{k}k\) is revolved about the line \(x = \frac{1}{k}\). Use cylindrical shells to find the volume of the resulting solid. (Assume \(k > 0\).)

33. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the center of a sphere. Show that the volume of the remaining solid depends only on the length \(L\) of the hole, not on the size of the sphere.

34. Use cylindrical shells to find the volume of the torus obtained by revolving the circle \(x^2 + y^2 = a^2\) about the line

FOCUS ON CONCEPTS

24. Let \(R_1\) and \(R_2\) be regions of the form shown in the accompanying figure. Use cylindrical shells to find a formula for the volume of the solid that results when (a) region \(R_1\) is revolved about the \(y\)-axis and (b) region \(R_2\) is revolved about the \(x\)-axis.

25. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve
\[ y = x^3 - 3x^2 + 2x \]
over \([0, 1]\) is revolved about the \(y\)-axis.

(b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.

26. Let \(f\) be continuous and nonnegative on \([a, b]\), and let \(R\) be the region that is enclosed by \(y = f(x)\) and \(y = 0\) for \(a \leq x \leq b\). Using the method of cylindrical shells, derive with explanation a formula for the volume of the solid generated by revolving \(R\) about the line \(x = k\), where \(k \leq a\).
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\[ x = b, \text{ where } b > a > 0. [\text{Hint: } \text{It may help in the integration to think of an integral as an area.}] \]

35. Let \( V_1 \) and \( V_2 \) be the volumes of the solids that result when the region enclosed by \( y = 1/x, y = 0, x = \frac{1}{2}, \text{ and } x = b \) \((b > \frac{1}{2})\) is revolved about the x-axis and y-axis, respectively. Is there a value of \( b \) for which \( V_1 = V_2 \)?

36. (a) Find the volume \( V \) of the solid generated when the region bounded by \( y = 1/(1 + x^4), y = 0, x = 1, \text{ and } x = b \) \((b > 1)\) is revolved about the y-axis.

(b) Find \( \lim_{b \to +} V \).

37. Writing Faced with the problem of computing the volume of a solid of revolution, how would you go about deciding whether to use the method of disks/washers or the method of cylindrical shells?

38. Writing With both the method of disks/washers and with the method of cylindrical shells, we integrate an “area” to get the volume of a solid of revolution. However, these two approaches differ in very significant ways. Write a brief paragraph that discusses these differences.

✔ QUICK CHECK ANSWERS 6.3

1. (a) \( 2\pi x(1 + \sqrt{x}) \) (b) \( \int_1^4 2\pi x(1 + \sqrt{x}) \, dx \)

2. (a) \( 2\pi(5 - x)(1 + \sqrt{x}) \) (b) \( \int_1^4 2\pi(5 - x)(1 + \sqrt{x}) \, dx \)

3. \( \int_0^4 2\pi y[4 - (y - 2)^2] \, dy \)

6.4 LENGTH OF A PLANE CURVE

In this section we will use the tools of calculus to study the problem of finding the length of a plane curve.

ARC LENGTH

Our first objective is to define what we mean by the length (also called the arc length) of a plane curve \( y = f(x) \) over an interval \([a, b]\) (Figure 6.4.1). Once that is done we will be able to focus on the problem of computing arc lengths. To avoid some complications that would otherwise occur, we will impose the requirement that \( f' \) be continuous on \([a, b]\), in which case we will say that \( y = f(x) \) is a smooth curve on \([a, b]\) or that \( f \) is a smooth function on \([a, b]\). Thus, we will be concerned with the following problem.

6.4.1 ARC LENGTH PROBLEM Suppose that \( y = f(x) \) is a smooth curve on the interval \([a, b]\). Define and find a formula for the arc length \( L \) of the curve \( y = f(x) \) over the interval \([a, b]\).

To define the arc length of a curve we start by breaking the curve into small segments. Then we approximate the curve segments by line segments and add the lengths of the line segments to form a Riemann sum. Figure 6.4.2 illustrates how such line segments tend to become better and better approximations to a curve as the number of segments increases. As the number of segments increases, the corresponding Riemann sums approach a definite integral whose value we will take to be the arc length \( L \) of the curve.

To implement our idea for solving Problem 6.4.1, divide the interval \([a, b]\) into \( n \) subintervals by inserting points \( x_1, x_2, \ldots, x_{n-1} \) between \( a = x_0 \) and \( b = x_n \). As shown in Figure 6.4.3a, let \( P_0, P_1, \ldots, P_n \) be the points on the curve with x-coordinates \( a = x_0, \ldots, \)
6.4 Length of a Plane Curve

Figure 6.4.2

Shorter line segments provide a better approximation to the curve.

Figure 6.4.3

(a) $x_1, x_2, \ldots, x_{n-1}, b = x_n$ and join these points with straight line segments. These line segments form a **polygonal path** that we can regard as an approximation to the curve $y = f(x)$.

As indicated in Figure 6.4.3b, the length $L_k$ of the $k$th line segment in the polygonal path is

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \quad (1)$$

If we now add the lengths of these line segments, we obtain the following approximation to the length $L$ of the curve

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \quad (2)$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point $x^*_k$ between $x_{k-1}$ and $x_k$ such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x^*_k) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x^*_k) \Delta x_k$$

and hence we can rewrite (2) as

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(x^*_k)]^2(\Delta x_k)^2} = \sum_{k=1}^{n} \sqrt{1 + [f'(x^*_k)]^2} \Delta x_k$$

Thus, taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the following integral that defines the arc length $L$:

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x^*_k)]^2} \Delta x_k = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx$$

In summary, we have the following definition.
6.4.2 **Definition** If \( y = f(x) \) is a smooth curve on the interval \([a, b]\), then the arc length \( L \) of this curve over \([a, b]\) is defined as

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx
\]

(3)

This result provides both a definition and a formula for computing arc lengths. Where convenient, (3) can also be expressed as

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

(4)

Moreover, for a curve expressed in the form \( x = g(y) \), where \( g' \) is continuous on \([c, d]\), the arc length \( L \) from \( y = c \) to \( y = d \) can be expressed as

\[
L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy
\]

(5)

**Example 1** Find the arc length of the curve \( y = x^{3/2} \) from \((1, 1)\) to \((2, 2\sqrt{2})\) (Figure 6.4.4) in two ways: (a) using Formula (4) and (b) using Formula (5).

**Solution (a).**

\[
\frac{dy}{dx} = \frac{3}{2} x^{1/2}
\]

and since the curve extends from \( x = 1 \) to \( x = 2 \), it follows from (4) that

\[
L = \int_1^2 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2} \, dx = \int_1^2 \sqrt{1 + \left(\frac{9}{4} x\right)} \, dx
\]

To evaluate this integral we make the \( u \)-substitution

\[
u = 1 + \frac{9}{4} x, \quad dv = \frac{9}{4} \, dx
\]

and then change the \( x \)-limits of integration \((x = 1, x = 2)\) to the corresponding \( u \)-limits \((u = 1 + \frac{9}{4}, u = \frac{22}{4})\):

\[
L = \frac{4}{9} \int_{13/4}^{22/4} u^{1/2} \, du = \frac{8}{27} \left[ u^{3/2} \right]_{13/4}^{22/4} = \frac{8}{27} \left[ \left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2} \right]
\]

\[
= \frac{22\sqrt{22} - 13\sqrt{13}}{27} \approx 2.09
\]

**Solution (b).** To apply Formula (5) we must first rewrite the equation \( y = x^{3/2} \) so that \( x \) is expressed as a function of \( y \). This yields \( x = y^{2/3} \) and

\[
\frac{dx}{dy} = \frac{2}{3} y^{-1/3}
\]

Since the curve extends from \( y = 1 \) to \( y = 2\sqrt{2} \), it follows from (5) that

\[
L = \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9} y^{-2/3}} \, dy = \frac{1}{3} \int_1^{2\sqrt{2}} y^{-1/3} \sqrt{9 y^{2/3} + 4} \, dy
\]
6.4 Length of a Plane Curve

To evaluate this integral we make the \( u \)-substitution
\[
u = 9y^{2/3} + 4, \quad du = 6y^{-1/3} \, dy
\]
and change the \( y \)-limits of integration \((y = 1, y = 2\sqrt{2})\) to the corresponding \( u \)-limits \((u = 13, u = 22)\). This gives
\[
L = \frac{1}{18} \int_{13}^{22} u^{1/2} \, du = \frac{1}{27} u^{3/2} \left|_{13}^{22} \right. = \frac{1}{27} (22)^{3/2} - (13)^{3/2} = \frac{22\sqrt{22} - 13\sqrt{13}}{27}
\]
The answer in part (b) agrees with that in part (a); however, the integration in part (b) is more tedious. In problems where there is a choice between using (4) or (5), it is often the case that one of the formulas leads to a simpler integral than the other. ◀

FINDING ARC LENGTH BY NUMERICAL METHODS

In the next chapter we will develop some techniques of integration that will enable us to find exact values of more integrals encountered in arc length calculations; however, generally speaking, most such integrals are impossible to evaluate in terms of elementary functions. In these cases one usually approximates the integral using a numerical method such as the midpoint rule discussed in Section 5.4.

Example 2

From (4), the arc length of \( y = \sin x \) from \( x = 0 \) to \( x = \pi \) is given by the integral
\[
L = \int_{0}^{\pi} \sqrt{1 + (\cos x)^2} \, dx
\]
This integral cannot be evaluated in terms of elementary functions; however, using a calculating utility with a numerical integration capability yields the approximation \( L \approx 3.8202 \). ◀

Quick Check Exercises 6.4

1. A function \( f \) is smooth on \([a, b]\) if \( f' \) is _________ on \([a, b]\).
2. If a function \( f \) is smooth on \([a, b]\), then the length of the curve \( y = f(x) \) over \([a, b]\) is _________.
3. The distance between points \((1, 0)\) and \((e, 1)\) is _________.

Exercise Set 6.4

1. Use the Theorem of Pythagoras to find the length of the line segment \( y = 2x \) from \((1, 2)\) to \((2, 4)\), and confirm that the value is consistent with the length computed using
   (a) Formula (4)  (b) Formula (5).
2. Use the Theorem of Pythagoras to find the length of the line segment \( y = 5x \) from \((0, 0)\) and \((1, 5)\), and confirm that the value is consistent with the length computed using
   (a) Formula (4)  (b) Formula (5).
3–8 Find the exact arc length of the curve over the interval.
4. \( x = \frac{1}{3}(y^2 + 2)^{3/2} \) from \( y = 0 \) to \( y = 1 \)
5. \( y = x^{2/3} \) from \( x = 1 \) to \( x = 8 \)
6. \( y = (x^6 + 8)/(16x^7) \) from \( x = 2 \) to \( x = 3 \)
7. \( 24xy = y^4 + 48 \) from \( y = 2 \) to \( y = 4 \)
8. \( x = \frac{1}{2}y^4 + \frac{1}{2}y^{-2} \) from \( y = 1 \) to \( y = 4 \)

9–12 True–False

Determine whether the statement is true or false. Explain your answer.

9. The graph of \( y = \sqrt{1 - x^2} \) is a smooth curve on \([-1, 1]\).
10. The approximation
\[
L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}
\]
for arc length is not expressed in the form of a Riemann sum.

11. The approximation
\[
L \approx \sum_{k=1}^{n} \sqrt{1 + [f'(x_k)]^2} \Delta x_k
\]
for arc length is exact when \( f \) is a linear function of \( x \).

12. In our definition of the arc length for the graph of \( y = f(x) \), we need \( f'(x) \) to be a continuous function in order for \( f \) to satisfy the hypotheses of the Mean-Value Theorem (4.8.2).

\[\text{FOCUS ON CONCEPTS}\]
13. \( y = \ln(\sec x) \) from \( x = 0 \) to \( x = \pi/4 \)
14. \( y = \ln(\sin x) \) from \( x = \pi/4 \) to \( x = \pi/2 \)

15. Consider the curve \( y = x^{2/3} \).
   (a) Sketch the portion of the curve between \( x = -1 \) and \( x = 8 \).
   (b) Explain why Formula (4) cannot be used to find the arc length of the curve sketched in part (a).
   (c) Find the arc length of the curve sketched in part (a).

16. The curve segment \( y = x^2 \) from \( x = 1 \) to \( x = 2 \) may also be expressed as the graph of \( x = \sqrt[3]{y} \) from \( y = 1 \) to \( y = 4 \). Set up two integrals that give the arc length of this curve segment, one by integrating with respect to \( x \), and the other by integrating with respect to \( y \). Demonstrate a substitution that verifies that these two integrals are equal.

17. Consider the curve segments \( y = x^2 \) from \( x = 1/3 \) to \( x = 2 \) and \( y = \sqrt{x} \) from \( x = 1/4 \) to \( x = 4 \).
   (a) Graph the two curve segments and use your graphs to explain why the lengths of these two curve segments should be equal.
   (b) Set up integrals that give the arc lengths of the curve segments by integrating with respect to \( x \). Demonstrate a substitution that verifies that these two integrals are equal.
   (c) Set up integrals that give the arc lengths of the curve segments by integrating with respect to \( y \).
   (d) Approximate the arc length of each curve segment using Formula (2) with \( n = 10 \) equal subintervals.
   (e) Which of the two approximations in part (d) is more accurate? Explain.
   (f) Use the midpoint approximation with \( n = 10 \) subintervals to approximate each arc length integral in part (b).

18. Follow the directions of Exercise 17 for the curve segments \( y = x^{8/3} \) from \( x = 10^{-3} \) to \( x = 1 \) and \( y = x^{3/8} \) from \( x = 10^{-8} \) to \( x = 1 \).

19. Follow the directions of Exercise 17 for the curve segment \( y = \tan x \) from \( x = 0 \) to \( x = \pi/3 \) and for the curve segment \( y = \tan^{-1} x \) from \( x = 0 \) to \( x = \sqrt{3} \).

20. Let \( y = f(x) \) be a smooth curve on the closed interval \([a, b]\). Prove that if \( m \) and \( M \) are nonnegative numbers such that \( m \leq |f'(x)| \leq M \) for all \( x \) in \([a, b]\), then the arc length \( L \) of \( y = f(x) \) over the interval \([a, b]\) satisfies the inequalities
\[
(b - a)\sqrt{1 + m^2} \leq L \leq (b - a)\sqrt{1 + M^2}
\]

21. Use the result of Exercise 20 to show that the arc length \( L \) of \( y = \sec x \) over the interval \( 0 \leq x \leq \pi/3 \) satisfies
\[
\frac{\pi}{3} \leq L \leq \frac{\pi}{3}\sqrt{13}
\]

\[\text{FOCUS ON CONCEPTS}\]
22. A basketball player makes a successful shot from the free throw line. Suppose that the path of the ball from the moment of release to the moment it enters the hoop is described by
\[
y = 2.15 + 2.09x - 0.41x^2, \quad 0 \leq x \leq 4.6
\]
where \( x \) is the horizontal distance (in meters) from the point of release, and \( y \) is the vertical distance (in meters) above the floor. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance the ball travels from the moment it is released to the moment it enters the hoop. Round your answer to two decimal places.

23. The central span of the Golden Gate Bridge in California is 4200 ft long and is suspended from cables that rise 500 ft above the roadway on either side. Approximately how long is the portion of a cable that lies between the support towers on one side of the roadway? \([\text{Hint: As suggested by the accompanying figure, assume the cable is modeled by a parabola } y = ax^2 \text{ that passes through the point } (2100, 500)\]. Use a CAS or a calculating utility with numerical integration capability to approximate the length of the cable. Round your answer to the nearest foot.

\[\text{Figure Ex-23}\]
24. As shown in the accompanying figure, a horizontal beam with dimensions 2 in × 6 in × 16 ft is fixed at both ends and is subjected to a uniformly distributed load of 120 lb/ft. As a result of the load, the centerline of the beam undergoes a deflection that is described by

\[ y = -1.67 \times 10^{-3} (x^4 - 2Lx^3 + L^2x^2) \]

(0 ≤ x ≤ 192), where L = 192 in is the length of the unloaded beam, x is the horizontal distance along the beam measured in inches from the left end, and y is the deflection of the centerline in inches.

(a) Graph y versus x for 0 ≤ x ≤ 192.
(b) Find the maximum deflection of the centerline.
(c) Use a CAS or a calculator with a numerical integration capability to find the length of the centerline of the loaded beam. Round your answer to two decimal places.

![Figure Ex-24](image)

25. A golfer makes a successful chip shot to the green. Suppose that the path of the ball from the moment it is struck to the moment it hits the green is described by

\[ y = 12.54x - 0.41x^2 \]

where x is the horizontal distance (in yards) from the point where the ball is struck, and y is the vertical distance (in yards) above the fairway. Use a CAS or a calculating utility with a numerical integration capability to find the distance the ball travels from the moment it is struck to the moment it hits the green. Assume that the fairway and green are at the same level and round your answer to two decimal places.

26–34 These exercises assume familiarity with the basic concepts of parametric curves. If needed, an introduction to this material is provided in Web Appendix I.

26. Assume that no segment of the curve

\[ x = x(t), \quad y = y(t), \quad (a ≤ t ≤ b) \]

is traced more than once as t increases from a to b. Divide the interval [a, b] into n subintervals by inserting points \( t_1, t_2, \ldots, t_{n-1} \) between \( a = t_0 \) and \( b = t_n \). Let \( L \) denote the arc length of the curve. Give an informal argument for the approximation

\[ L \approx \sum_{k=1}^{n} \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2} \]

27–32 Use the arc length formula from Exercise 26 to find the arc length of the curve.

27. \( x = \frac{1}{2}t^3, \quad y = \frac{1}{2}t^2 \quad (0 ≤ t ≤ 1) \)
28. \( x = (1 + t)^2, \quad y = (1 + t)^3 \quad (0 ≤ t ≤ 1) \)
29. \( x = \cos 2t, \quad y = \sin 2t \quad (0 ≤ t ≤ π/2) \)
30. \( x = \cos t + t \sin t, \quad y = \sin t - t \cos t \quad (0 ≤ t ≤ π) \)
31. \( x = e^t \cos t, \quad y = e^t \sin t \quad (0 ≤ t ≤ π/2) \)
32. \( x = e^t (\sin t + \cos t), \quad y = e^t (\cos t - \sin t) \quad (1 ≤ t ≤ 4) \)

33. (a) Show that the total arc length of the ellipse

\[ x = 2 \cos t, \quad y = \sin t \quad (0 ≤ t ≤ 2π) \]

is given by

\[ 4 \int_{0}^{π/2} \sqrt{1 + 3 \sin^2 t} \, dt \]

(b) Use a CAS or a scientific calculator with a numerical integration capability to approximate the arc length in part (a). Round your answer to two decimal places.

(c) Suppose that the parametric equations in part (a) describe the path of a particle moving in the xy-plane, where t is time in seconds and x and y are in centimeters. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance traveled by the particle from \( t = 1.5 \) s to \( t = 4.8 \) s. Round your answer to two decimal places.

34. Show that the total arc length of the ellipse \( x = a \cos t, \quad y = b \sin t, \quad 0 ≤ t ≤ 2π \) for \( a > b > 0 \) is given by

\[ 4a \int_{0}^{π/2} \sqrt{1 - k^2 \cos^2 t} \, dt \]

where \( k = \sqrt{a^2 - b^2}/a \).

35. Writing In our discussion of Arc Length Problem 6.4.1, we derived the approximation

\[ L \approx \sum_{k=1}^{n} \sqrt{1 + [f'(x_k)]^2} \Delta x_k \]

Discuss the geometric meaning of this approximation. (Be sure to address the appearance of the derivative \( f' \).)

36. Writing Give examples in which Formula (4) for arc length cannot be applied directly, and describe how you would go about finding the arc length of the curve in each case. (Discuss both the use of alternative formulas and the use of numerical methods.)
6.5 AREA OF A SURFACE OF REVOLUTION

In this section we will consider the problem of finding the area of a surface that is generated by revolving a plane curve about a line.

SURFACE AREA

A surface of revolution is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter, and the lateral surface of a right circular cylinder can be generated by revolving a line segment about an axis that is parallel to it (Figure 6.5.1).

Some Surfaces of Revolution

In this section we will be concerned with the following problem.

6.5.1 SURFACE AREA PROBLEM Suppose that \( f \) is a smooth, nonnegative function on \([a, b]\) and that a surface of revolution is generated by revolving the portion of the curve \( y = f(x) \) between \( x = a \) and \( x = b \) about the \( x \)-axis (Figure 6.5.2). Define what is meant by the area \( S \) of the surface, and find a formula for computing it.

To motivate an appropriate definition for the area \( S \) of a surface of revolution, we will decompose the surface into small sections whose areas can be approximated by elementary formulas, add the approximations of the areas of the sections to form a Riemann sum that approximates \( S \), and then take the limit of the Riemann sums to obtain an integral for the exact value of \( S \).

To implement this idea, divide the interval \([a, b]\) into \( n \) subintervals by inserting points \( x_1, x_2, \ldots, x_{n-1} \) between \( a = x_0 \) and \( b = x_n \). As illustrated in Figure 6.5.3a, the corresponding points on the graph of \( f \) define a polygonal path that approximates the curve \( y = f(x) \) over the interval \([a, b]\). As illustrated in Figure 6.5.3b, when this polygonal path is revolved about the \( x \)-axis, it generates a surface consisting of \( n \) parts, each of which is a portion of a right circular cone called a frustum (from the Latin meaning “bit” or “piece”). Thus, the area of each part of the approximating surface can be obtained from the formula

\[
S = \pi(r_1 + r_2)L
\]

(1)

for the lateral area \( S \) of a frustum of slant height \( l \) and base radii \( r_1 \) and \( r_2 \) (Figure 6.5.4). As suggested by Figure 6.5.5, the \( k \)th frustum has radii \( f(x_{k-1}) \) and \( f(x_k) \) and height \( \Delta x_k \). Its slant height is the length \( L_k \) of the \( k \)th line segment in the polygonal path, which from Formula (1) of Section 6.4 is

\[
L_k = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}
\]
6.5 Area of a Surface of Revolution

This makes the lateral area $S_k$ of the $k$th frustum

$$S_k = \pi [f(x_{k-1}) + f(x_k)] \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

If we add these areas, we obtain the following approximation to the area $S$ of the entire surface:

$$S \approx \sum_{k=1}^{n} \pi [f(x_{k-1}) + f(x_k)] \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point $x_k^*$ between $x_{k-1}$ and $x_k$ such that

$$f(x_k) - f(x_{k-1}) = f'(x_k^*) \Delta x_k$$

and hence we can rewrite (2) as

$$S \approx \sum_{k=1}^{n} \pi [f(x_{k-1}) + f(x_k)] \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

However, this is not yet a Riemann sum because it involves the variables $x_{k-1}$ and $x_k$. To eliminate these variables from the expression, observe that the average value of the numbers $f(x_{k-1})$ and $f(x_k)$ lies between these numbers, so the continuity of $f$ and the Intermediate-Value Theorem (1.5.7) imply that there is a point $x_k^{**}$ between $x_{k-1}$ and $x_k$ such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) = f(x_{k-1}) + f'(x_k^*) \Delta x_k$$

Thus, (2) can be expressed as

$$S \approx \sum_{k=1}^{n} \pi [f(x_{k-1}) + f(x_k)] \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

Although this expression is close to a Riemann sum in form, it is not a true Riemann sum because it involves two variables $x_k^*$ and $x_k^{**}$, rather than $x_k^*$ alone. However, it is proved in advanced calculus courses that this has no effect on the limit because of the continuity of $f$. Thus, we can assume that $x_k^{**} = x_k^*$ when taking the limit, and this suggests that $S$ can be defined as

$$S = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \pi [f(x_k^*)] \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k = \int_{a}^{b} 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx$$

In summary, we have the following definition.
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6.5.2 Definition If \( f \) is a smooth, nonnegative function on \([a, b]\), then the surface area \( S \) of the surface of revolution that is generated by revolving the portion of the curve \( y = f(x) \) between \( x = a \) and \( x = b \) about the \( x \)-axis is defined as

\[
S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx
\]

This result provides both a definition and a formula for computing surface areas. Where convenient, this formula can also be expressed as

\[
S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\] (4)

Moreover, if \( g \) is nonnegative and \( x = g(y) \) is a smooth curve on the interval \([c, d]\), then the area of the surface that is generated by revolving the portion of a curve \( x = g(y) \) between \( y = c \) and \( y = d \) about the \( y \)-axis can be expressed as

\[
S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} \, dy = \int_c^d 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy
\] (5)

Example 1 Find the area of the surface that is generated by revolving the portion of the curve \( y = x^3 \) between \( x = 0 \) and \( x = 1 \) about the \( x \)-axis.

Solution. First sketch the curve; then imagine revolving it about the \( x \)-axis (Figure 6.5.6). Since \( y = x^3 \), we have \( dy/dx = 3x^2 \), and hence from (4) the surface area \( S \) is

\[
S = \int_0^1 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2} \, dx
\]

\[
= \int_0^1 2\pi x^3 \sqrt{1 + (9x^4)} \, dx = 2\pi \int_0^1 x^3 (1 + 9x^4)^{1/2} \, dx = \frac{2\pi}{36} \int_1^{10} u^{1/2} du
\]

\[
= \frac{2\pi}{36} \left[ \frac{2}{3} u^{3/2} \right]_1^{10} = \frac{2\pi}{27} (10^{3/2} - 1) \approx 3.56
\]

Example 2 Find the area of the surface that is generated by revolving the portion of the curve \( y = x^2 \) between \( x = 1 \) and \( x = 2 \) about the \( y \)-axis.

Solution. First sketch the curve; then imagine revolving it about the \( y \)-axis (Figure 6.5.7). Because the curve is revolved about the \( y \)-axis we will apply Formula (5). Toward this end, we rewrite \( y = x^2 \) as \( x = \sqrt{y} \) and observe that the \( y \)-values corresponding to \( x = 1 \) and
6.5 Area of a Surface of Revolution

Given curve about the

Use a CAS to find the exact area of the surface generated by revolving the portion of the curve \( y = f(x) \) between \( x = a \) and \( x = b \) about the \( x \)-axis is ________.

The lateral area of the frustum with slant height \( \sqrt{10} \) and base radii \( r_1 = 1 \) and \( r_2 = 2 \) is ________.

An integral expression for the area of the surface generated by revolving the line segment joining \((3,1)\) and \((6,2)\) about the \( x \)-axis is ________.

An integral expression for the area of the surface generated by revolving the line segment joining \((3,1)\) and \((6,2)\) about the \( y \)-axis is ________.

Quick Check Exercises 6.5

1. If \( f \) is a smooth, nonnegative function on \([a, b]\), then the surface area \( S \) of the surface of revolution generated by revolving the portion of the curve \( y = f(x) \) between \( x = a \) and \( x = b \) about the \( x \)-axis is ________.

2. The lateral area of the frustum with slant height \( \sqrt{10} \) and base radii \( r_1 = 1 \) and \( r_2 = 2 \) is ________.

3. An integral expression for the area of the surface generated by rotating the line segment joining \((3,1)\) and \((6,2)\) about the \( x \)-axis is ________.

4. An integral expression for the area of the surface generated by rotating the line segment joining \((3,1)\) and \((6,2)\) about the \( y \)-axis is ________.

Exercise Set 6.5

1–4 Find the area of the surface generated by revolving the given curve about the \( x \)-axis.

1. \( y = 3x, \ 0 \leq x \leq 1 \)

2. \( y = \sqrt{x}, \ 1 \leq x \leq 4 \)

3. \( y = \sqrt{4 - x^2}, \ -1 \leq x \leq 1 \)

4. \( y = \frac{3}{2}x, \ 1 \leq y \leq 8 \)

5–8 Find the area of the surface generated by revolving the given curve about the \( y \)-axis.

5. \( x = 9y + 1, \ 0 \leq y \leq 2 \)

6. \( x = y^3, \ 0 \leq y \leq 1 \)

7. \( x = \sqrt{9 - y^2}, \ -2 \leq y \leq 2 \)

8. \( x = 2\sqrt{1 - y}, \ -1 \leq y \leq 0 \)

9–12 Use a CAS to find the exact area of the surface generated by revolving the curve about the stated axis.

9. \( y = \sqrt{x} - \frac{1}{4}x^{3/2}, \ 1 \leq x \leq 3; \ x \)-axis

10. \( y = \frac{1}{4}x^3 + \frac{1}{4}x^{1/2}, \ 1 \leq x \leq 2; \ x \)-axis

11. \( 8xy^2 = 2y^3 + 1, \ 1 \leq y \leq 2; \ y \)-axis

12. \( x = \sqrt{16 - y}, \ 0 \leq y \leq 15; \ y \)-axis

13–16 Use a CAS or a calculating utility with a numerical integration capability to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places.

13. \( y = \sin x, \ 0 \leq x \leq \pi; \ x \)-axis

14. \( x = \tan y, \ 0 \leq y \leq \pi/4; \ y \)-axis

15. \( y = e^x, \ 0 \leq x \leq 1; \ x \)-axis

16. \( y = e^x, \ 1 \leq y \leq e; \ y \)-axis

17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. The lateral surface area \( S \) of a right circular cone with height \( h \) and base radius \( r \) is \( S = \pi r \sqrt{r^2 + h^2} \).

18. The lateral surface area of a frustum of slant height \( l \) and base radii \( r_1 \) and \( r_2 \) is equal to the lateral surface area of a right circular cylinder of height \( l \) and radius equal to the average of \( r_1 \) and \( r_2 \).
19. The approximation

\[ S \approx \sum_{k=1}^{n} 2\pi f(x_k^*) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k \]

for surface area is exact if \( f \) is a positive-valued constant function.

20. The expression

\[ \sum_{k=1}^{n} 2\pi f(x_k^*) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k \]

is not a true Riemann sum for

\[ \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx \]

21–22 Approximate the area of the surface using Formula (2) with \( n = 20 \) subintervals of equal width. Round your answer to two decimal places.


22. The surface of Exercise 16.

**FOCUS ON CONCEPTS**

23. Assume that \( y = f(x) \) is a smooth curve on the interval \([a, b]\) and assume that \( f(x) \geq 0 \) for \( a \leq x \leq b \). Derive a formula for the surface area generated when the curve \( y = f(x) \), \( a \leq x \leq b \), is revolved about the line \( y = -k \) \( (k > 0) \).

24. Would it be circular reasoning to use Definition 6.5.2 to find the surface area of a frustum of a right circular cone? Explain your answer.

25. Show that the area of the surface of a sphere of radius \( r \) is \( 4\pi r^2 \). [Hint: Revolve the semicircle \( y = \sqrt{r^2 - x^2} \) about the \( x \)-axis.]

26. The accompanying figure shows a spherical cap of height \( h \) cut from a sphere of radius \( r \). Show that the surface area \( S \) of the cap is \( S = 2\pi rh \). [Hint: Revolve an appropriate portion of the circle \( x^2 + y^2 = r^2 \) about the \( y \)-axis.]

27. The portion of a sphere that is cut by two parallel planes is called a **zone**. Use the result of Exercise 26 to show that the surface area of a zone depends on the radius of the sphere and the distance between the planes, but not on the location of the zone.

28. Let \( y = f(x) \) be a smooth curve on the interval \([a, b]\) and assume that \( f(x) \geq 0 \) for \( a \leq x \leq b \). By the Extreme-Value Theorem (4.4.2), the function \( f \) has a maximum value \( K \) and a minimum value \( k \) on \([a, b]\). Prove: If \( L \) is the arc length of the curve \( y = f(x) \) between \( x = a \) and \( x = b \), and if \( S \) is the area of the surface that is generated by revolving this curve about the \( x \)-axis, then

\[ 2\pi k L \leq S \leq 2\pi KL \]

29. Use the results of Exercise 28 above and Exercise 21 in Section 6.4 to show that the area \( S \) of the surface generated by revolving the curve \( y = \sec x \), \( 0 \leq x \leq \pi/3 \), about the \( x \)-axis satisfies

\[ \frac{2\pi^2}{3} \leq S \leq \frac{4\pi^2}{3} \sqrt{13} \]

30. Let \( y = f(x) \) be a smooth curve on \([a, b]\) and assume that \( f(x) \geq 0 \) for \( a \leq x \leq b \). Let \( A \) be the area under the curve \( y = f(x) \) between \( x = a \) and \( x = b \), and let \( S \) be the area of the surface obtained when this section of curve is revolved about the \( x \)-axis.

(a) Prove that \( 2\pi A \leq S \).

(b) For what functions \( f \) is \( 2\pi A = S \)?

31–37 These exercises assume familiarity with the basic concepts of parametric curves. If needed, an introduction to this material is provided in Web Appendix I.

31–32 For these exercises, divide the interval \([a, b]\) into \( n \) subintervals by inserting points \( t_1, t_2, \ldots, t_{n-1} \) between \( a = t_0 \) and \( b = t_n \), and assume that \( x(t) \) and \( y(t) \) are continuous functions and that no segment of the curve

\[ x = x(t), \quad y = y(t) \quad (a \leq t \leq b) \]

is traced more than once.

31. Let \( S \) be the area of the surface generated by revolving the curve \( x = x(t), \quad y = y(t) \) \( (a \leq t \leq b) \) about the \( x \)-axis. Explain how \( S \) can be approximated by

\[ S \approx \sum_{k=1}^{n} (\pi[y(t_{k+1}) - y(t_k)] \times \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2}) \]

Using results from advanced calculus, it can be shown that as \( \Delta t_k \to 0 \), this sum converges to

\[ S = \int_a^b 2\pi y(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt \]  \hspace{1cm} (A)

32. Let \( S \) be the area of the surface generated by revolving the curve \( x = x(t), \quad y = y(t) \) \( (a \leq t \leq b) \) about the \( y \)-axis. Explain how \( S \) can be approximated by

\[ S \approx \sum_{k=1}^{n} (\pi[x(t_{k+1}) + x(t_k)] \times \sqrt{[x(t_k) - x(t_{k-1})]^2 + [y(t_k) - y(t_{k-1})]^2}) \]

Using results from advanced calculus, it can be shown that as \( \Delta t_k \to 0 \), this sum converges to

\[ S = \int_a^b 2\pi x(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt \]  \hspace{1cm} (B)
33–37 Use Formulas (A) and (B) from Exercises 31 and 32.

33. Find the area of the surface generated by revolving the parametric curve \( x = t^2, \ y = 2t \) \((0 \leq t \leq 4)\) about the \( x \)-axis.

34. Use a CAS to find the area of the surface generated by revolving the parametric curve 
\[
x = \cos^2 t, \quad y = 5 \sin t \quad (0 \leq t \leq \pi/2)
\]
about the \( x \)-axis.

35. Find the area of the surface generated by revolving the parametric curve 
\[
x = t, \quad y = 2t^2 \quad (0 \leq t \leq 1)\]
about the \( y \)-axis.

36. Find the area of the surface generated by revolving the parametric curve 
\[
x = \cos^2 t, \quad y = \sin^2 t \quad (0 \leq t \leq \pi/2)\]
about the \( y \)-axis.

37. By revolving the semicircle 
\[
x = r \cos t, \quad y = r \sin t \quad (0 \leq t \leq \pi)
\]
about the \( x \)-axis, show that the surface area of a sphere of radius \( r \) is \( 4\pi r^2 \).

38. Writing Compare the derivation of Definition 6.5.2 with that of Definition 6.4.2. Discuss the geometric features that result in similarities in the two definitions.

39. Writing Discuss what goes wrong if we replace the frustums of right circular cones by right circular cylinders in the derivation of Definition 6.5.2.

Quick Check Answers 6.5

1. \[ \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx \]
2. \[ 3\sqrt{10} \pi \]
3. \[ \int_3^6 (2\pi) \left( \frac{x}{3} \right) \sqrt{\frac{10}{9}} \, dx = \int_3^6 \frac{2\sqrt{10} \pi}{9} \, dx \]
4. \[ \int_1^2 (2\pi)(3y) \sqrt{10} \, dy \]

6.6 Work

In this section we will use the integration tools developed in the preceding chapter to study some of the basic principles of “work,” which is one of the fundamental concepts in physics and engineering.

The Role of Work in Physics and Engineering

In this section we will be concerned with two related concepts, work and energy. To put these ideas in a familiar setting, when you push a stalled car for a certain distance you are performing work, and the effect of your work is to make the car move. The energy of motion caused by the work is called the kinetic energy of the car. The exact connection between work and kinetic energy is governed by a principle of physics called the work–energy relationship. Although we will touch on this idea in this section, a detailed study of the relationship between work and energy will be left for courses in physics and engineering. Our primary goal here will be to explain the role of integration in the study of work.

Work Done by a Constant Force Applied in the Direction of Motion

When a stalled car is pushed, the speed that the car attains depends on the force \( F \) with which it is pushed and the distance \( d \) over which that force is applied (Figure 6.6.1). Force and distance appear in the following definition of work.
6.6.1 Definition

If a constant force of magnitude \( F \) is applied in the direction of motion of an object, and if that object moves a distance \( d \), then we define the work \( W \) performed by the force on the object to be

\[
W = F \cdot d
\]  \hspace{1cm} (1)

Common units for measuring force are newtons (N) in the International System of Units (SI), dynes (dyn) in the centimeter-gram-second (CGS) system, and pounds (lb) in the British Engineering (BE) system. One newton is the force required to give a mass of 1 kg an acceleration of \( 1 \text{ m/s}^2 \), one dyne is the force required to give a mass of 1 g an acceleration of \( 1 \text{ cm/s}^2 \), and one pound of force is the force required to give a mass of 1 slug an acceleration of \( 1 \text{ ft/s}^2 \).

It follows from Definition 6.6.1 that work has units of force times distance. The most common units of work are newton-meters (N·m), dyne-centimeters (dyn·cm), and foot-pounds (ft·lb). As indicated in Table 6.6.1, one newton-meter is also called a joule (J), and one dyne-centimeter is also called an erg. One foot-pound is approximately 1.36 J.

Table 6.6.1

<table>
<thead>
<tr>
<th>SYSTEM</th>
<th>FORCE</th>
<th>×</th>
<th>DISTANCE</th>
<th>=</th>
<th>WORK</th>
</tr>
</thead>
<tbody>
<tr>
<td>SI</td>
<td>newton (N)</td>
<td>meter (m)</td>
<td></td>
<td>joule (J)</td>
<td></td>
</tr>
<tr>
<td>CGS</td>
<td>dyne (dyn)</td>
<td>centimeter (cm)</td>
<td></td>
<td>erg</td>
<td></td>
</tr>
<tr>
<td>BE</td>
<td>pound (lb)</td>
<td>foot (ft)</td>
<td></td>
<td>foot-pound (ft·lb)</td>
<td></td>
</tr>
</tbody>
</table>

Conversion factors:

\[1 \text{ N} = 10^5 \text{ dyn} \approx 0.225 \text{ lb}\]
\[1 \text{ lb} \approx 4.45 \text{ N}\]
\[1 \text{ J} = 10^7 \text{ erg} \approx 0.738 \text{ ft} \cdot \text{lb}\]
\[1 \text{ ft} \cdot \text{lb} \approx 1.36 \text{ J}\]

Example 1

An object moves 5 ft along a line while subjected to a constant force of 100 lb in its direction of motion. The work done is

\[W = F \cdot d = 100 \cdot 5 = 500 \text{ ft·lb}\]

An object moves 25 m along a line while subjected to a constant force of 4 N in its direction of motion. The work done is

\[W = F \cdot d = 4 \cdot 25 = 100 \text{ N·m} = 100 \text{ J}\]

Example 2

In the 1976 Olympics, Vasili Alexeev astounded the world by lifting a record-breaking 562 lb from the floor to above his head (about 2 m). Equally astounding was the feat of strongman Paul Anderson, who in 1957 braced himself on the floor and used his back to lift 6270 lb of lead and automobile parts a distance of 1 cm. Who did more work?

Solution. To lift an object one must apply sufficient force to overcome the gravitational force that the Earth exerts on that object. The force that the Earth exerts on an object is that object’s weight; thus, in performing their feats, Alexeev applied a force of 562 lb over a distance of 2 m and Anderson applied a force of 6270 lb over a distance of 1 cm. Pounds are units in the BE system, meters are units in SI, and centimeters are units in the CGS system. We will need to decide on the measurement system we want to use and be consistent. Let us agree to use SI and express the work of the two men in joules. Using the conversion factor in Table 6.6.1 we obtain

\[562 \text{ lb} \approx 562 \text{ lb} \times 4.45 \text{ N/lb} \approx 2500 \text{ N}\]
\[6270 \text{ lb} \approx 6270 \text{ lb} \times 4.45 \text{ N/lb} \approx 27,900 \text{ N}\]
Using these values and the fact that $1 \text{ cm} = 0.01 \text{ m}$ we obtain

Alexeev’s work $= (2500 \text{ N}) \times (2 \text{ m}) = 5000 \text{ J}$

Anderson’s work $= (27900 \text{ N}) \times (0.01 \text{ m}) = 279 \text{ J}$

Therefore, even though Anderson’s lift required a tremendous upward force, it was applied over such a short distance that Alexeev did more work.

WORK DONE BY A VARIABLE FORCE APPLIED IN THE DIRECTION OF MOTION

Many important problems are concerned with finding the work done by a variable force that is applied in the direction of motion. For example, Figure 6.6.2a shows a spring in its natural state (neither compressed nor stretched). If we want to pull the block horizontally (Figure 6.6.2b), then we would have to apply more and more force to the block to overcome the increasing force of the stretching spring. Thus, our next objective is to define what is meant by the work performed by a variable force and to find a formula for computing it. This will require calculus.

6.6.2 PROBLEM Suppose that an object moves in the positive direction along a coordinate line while subjected to a variable force $F(x)$ that is applied in the direction of motion. Define what is meant by the work $W$ performed by the force on the object as the object moves from $x = a$ to $x = b$, and find a formula for computing the work.

The basic idea for solving this problem is to break up the interval $[a, b]$ into subintervals that are sufficiently small that the force does not vary much on each subinterval. This will allow us to treat the force as constant on each subinterval and to approximate the work on each subinterval using Formula (1). By adding the approximations to the work on the subintervals, we will obtain a Riemann sum that approximates the work $W$ over the entire interval, and by taking the limit of the Riemann sums we will obtain an integral for $W$.

To implement this idea, divide the interval $[a, b]$ into $n$ subintervals by inserting points $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. We can use Formula (1) to approximate the work $W_k$ done in the $k$th subinterval by choosing any point $x^*_k$ in this interval and regarding the force to have a constant value $F(x^*_k)$ throughout the interval. Since the width of the $k$th subinterval is $x_k - x_{k-1} = \Delta x_k$, this yields the approximation

$$W_k \approx F(x^*_k)\Delta x_k$$

Adding these approximations yields the following Riemann sum that approximates the work $W$ done over the entire interval:

$$W \approx \sum_{k=1}^{n} F(x^*_k)\Delta x_k$$

Taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the definite integral

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} F(x^*_k)\Delta x_k = \int_{a}^{b} F(x) \, dx$$

In summary, we have the following result.
6.6.3 Definition Suppose that an object moves in the positive direction along a coordinate line over the interval \([a, b]\) while subjected to a variable force \(F(x)\) that is applied in the direction of motion. Then we define the work \(W\) performed by the force on the object to be

\[
W = \int_a^b F(x) \, dx
\]  

(2)

Hooke’s law [Robert Hooke (1635–1703), English physicist] states that under appropriate conditions a spring that is stretched \(x\) units beyond its natural length pulls back with a force

\[ F(x) = kx \]

where \(k\) is a constant (called the spring constant or spring stiffness). The value of \(k\) depends on such factors as the thickness of the spring and the material used in its composition. Since \(k = F(x)/x\), the constant \(k\) has units of force per unit length.

Example 3 A spring exerts a force of 5 N when stretched 1 m beyond its natural length.

(a) Find the spring constant \(k\).

(b) How much work is required to stretch the spring 1.8 m beyond its natural length?

Solution (a). From Hooke’s law,

\[ F(x) = kx \]

From the data, \(F(x) = 5\) N when \(x = 1\) m, so \(5 = k \cdot 1\). Thus, the spring constant is \(k = 5\) newtons per meter (N/m). This means that the force \(F(x)\) required to stretch the spring \(x\) meters is

\[ F(x) = 5x \]  

(3)

Solution (b). Place the spring along a coordinate line as shown in Figure 6.6.3. We want to find the work \(W\) required to stretch the spring over the interval from \(x = 0\) to \(x = 1.8\). From (2) and (3) the work \(W\) required is

\[
W = \int_a^b F(x) \, dx = \int_0^{1.8} 5x \, dx = \frac{5x^2}{2} \bigg|_0^{1.8} = 8.1 \text{ J} \]

Example 4 An astronaut’s weight (or more precisely, Earth weight) is the force exerted on the astronaut by the Earth’s gravity. As the astronaut moves upward into space, the gravitational pull of the Earth decreases, and hence so does his or her weight. If the Earth is assumed to be a sphere of radius 4000 mi, then it follows from Newton’s Law of Universal Gravitation that an astronaut who weighs 150 lb on Earth will have a weight of

\[
w(x) = \frac{2,400,000,000}{x^2} \text{ lb, } x \geq 4000
\]

at a distance of \(x\) miles from the Earth’s center (Exercise 25). Use this formula to estimate the work in foot-pounds required to lift the astronaut 220 miles upward to the International Space Station.

Solution. Since the Earth has a radius of 4000 mi, the astronaut is lifted from a point that is 4000 mi from the Earth’s center to a point that is 4220 mi from the Earth’s center. Thus,
from (2), the work $W$ required to lift the astronaut is

$$W = \int_{4000}^{4220} \frac{2,400,000,000}{x^2} \, dx\]$$

$$= -\frac{2,400,000,000}{x} \bigg|_{4000}^{4220}$$

$$\approx -568,720 + 600,000$$

$$= 31,280 \text{ mile-pounds}$$

$$= (31,280 \text{ mi-lb}) \times (5280 \text{ ft/mi})$$

$$\approx 1.65 \times 10^8 \text{ ft-lb} \uparrow$$

---

### 6.6 Work

#### CALCULATING WORK FROM BASIC PRINCIPLES

Some problems cannot be solved by mechanically substituting into formulas, and one must return to basic principles to obtain solutions. This is illustrated in the next example.

**Example 5** Figure 6.6.4 shows a conical container of radius 10 ft and height 30 ft. Suppose that this container is filled with water to a depth of 15 ft. How much work is required to pump all of the water out through a hole in the top of the container?

**Solution.** Our strategy will be to divide the water into thin layers, approximate the work required to move each layer to the top of the container, add the approximations for the layers to obtain a Riemann sum that approximates the total work, and then take the limit of the Riemann sums to produce an integral for the total work.

To implement this idea, introduce an $x$-axis as shown in Figure 6.6.4a, and divide the water into $n$ layers with $\Delta x_k$ denoting the thickness of the $k$th layer. This division induces a partition of the interval $[15, 30]$ into $n$ subintervals. Although the upper and lower surfaces of the $k$th layer are at different distances from the top, the difference will be small if the layer is thin, and we can reasonably assume that the entire layer is concentrated at a single point $x^*_k$ (Figure 6.6.4a). Thus, the work $W_k$ required to move the $k$th layer to the top of the container is approximately

$$W_k \approx F_k x^*_k$$

(4)

where $F_k$ is the force required to lift the $k$th layer. But the force required to lift the $k$th layer is the force needed to overcome gravity, and this is the same as the weight of the layer. If the layer is very thin, we can approximate the volume of the $k$th layer with the volume of a cylinder of height $\Delta x_k$ and radius $r_k$, where (by similar triangles)

$$\frac{r_k}{x^*_k} = \frac{10}{30} = \frac{1}{3}$$

or, equivalently, $r_k = x^*_k/3$ (Figure 6.6.4b). Therefore, the volume of the $k$th layer of water is approximately

$$\pi r_k^2 \Delta x_k = \pi (x^*_k)^2 \Delta x_k = \frac{\pi}{9} (x^*_k)^2 \Delta x_k$$

Since the weight density of water is 62.4 lb/ft$^3$, it follows that

$$F_k \approx \frac{62.4\pi}{9} (x^*_k)^2 \Delta x_k$$

Thus, from (4)

$$W_k \approx \left( \frac{62.4\pi}{9} (x^*_k)^2 \right) x^*_k \Delta x_k \approx \frac{62.4\pi}{9} (x^*_k)^3 \Delta x_k$$
and hence the work $W$ required to move all $n$ layers has the approximation

$$W = \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} \frac{62.4\pi}{9} (x_k^*)^3 \Delta x_k$$

To find the exact value of the work we take the limit as $\max \Delta x_k \to 0$. This yields

$$W = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \frac{62.4\pi}{9} (x_k^*)^3 \Delta x_k = \int_{15}^{30} \frac{62.4\pi}{9} x^3 \, dx$$

$$= \frac{62.4\pi}{9} \left( \frac{x^4}{4} \right)_{15}^{30} = 1,316,250\pi \approx 4,135,000 \text{ ft} \cdot \text{lb}$$

\[\text{Figure 6.6.4}\]

\section*{THE WORK–ENERGY RELATIONSHIP}

When you see an object in motion, you can be certain that somehow work has been expended to create that motion. For example, when you drop a stone from a building, the stone gathers speed because the force of the Earth’s gravity is performing work on it, and when a hockey player strikes a puck with a hockey stick, the work performed on the puck during the brief period of contact with the stick creates the enormous speed of the puck across the ice. However, experience shows that the speed obtained by an object depends not only on the amount of work done, but also on the mass of the object. For example, the work required to throw a 5 oz baseball 50 mi/h would accelerate a 10 lb bowling ball to less than 9 mi/h.

Using the method of substitution for definite integrals, we will derive a simple equation that relates the work done on an object to the object’s mass and velocity. Furthermore, this equation will allow us to motivate an appropriate definition for the “energy of motion” of an object. As in Definition 6.6.3, we will assume that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a force $F(x)$ that is applied in the direction of motion. We let $m$ denote the mass of the object, and we let $x = x(t), v = v(t) = x'(t),$ and $a = a(t) = v'(t)$ denote the respective position, velocity, and acceleration of the object at time $t$. We will need the following important result from physics that relates the force acting on an object with the mass and acceleration of the object.

\begin{center}
\begin{tabular}{|c|}
\hline
6.6.4 NEWTON’S SECOND LAW OF MOTION & If an object with mass $m$ is subjected to a force $F$, then the object undergoes an acceleration $a$ that satisfies the equation \\
& $F = ma$ \\
\hline
\end{tabular}
\end{center}

It follows from Newton’s Second Law of Motion that

$$F(x(t)) = ma(t) = mv'(t)$$
Assume that 

\[ x(t_0) = a \quad \text{and} \quad x(t_1) = b \]

with 

\[ v(t_0) = v_i \quad \text{and} \quad v(t_1) = v_f \]

the initial and final velocities of the object, respectively. Then

\[ W = \int_a^b F(x) \, dx = \int_{x(t_0)}^{x(t_1)} F(x) \, dx \]

By Theorem 5.9.1 with \( x = x(t), dx = x'(t) \, dt \)

\[ = \int_{t_0}^{t_1} mv'(t) \, v(t) \, dt = \int_{t_0}^{t_1} mv(t) \, v'(t) \, dt \]

By Theorem 5.9.1 with \( v = v(t), dv = v'(t) \, dt \)

\[ = \int_{v(t_0)}^{v(t_1)} mv \, dv = \frac{1}{2}mv_f^2|_{v_i} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \]

We see from the equation

\[ W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \quad (6) \]

that the work done on the object is equal to the change in the quantity \( \frac{1}{2}mv^2 \) from its initial value to its final value. We will refer to Equation (6) as the work–energy relationship. If we define the “energy of motion” or kinetic energy of our object to be given by

\[ K = \frac{1}{2}mv^2 \quad (7) \]

then Equation (6) tells us that the work done on an object is equal to the change in the object’s kinetic energy. Loosely speaking, we may think of work done on an object as being “transformed” into kinetic energy of the object. The units of kinetic energy are the same as the units of work. For example, in SI kinetic energy is measured in joules (J).

**Example 6** A space probe of mass \( m = 5.00 \times 10^4 \) kg travels in deep space subjected only to the force of its own engine. Starting at a time when the speed of the probe is \( v = 1.10 \times 10^4 \) m/s, the engine is fired continuously over a distance of \( 2.50 \times 10^6 \) m with a constant force of \( 4.00 \times 10^5 \) N in the direction of motion. What is the final speed of the probe?

**Solution.** Since the force applied by the engine is constant and in the direction of motion, the work \( W \) expended by the engine on the probe is

\[ W = \text{force} \times \text{distance} = (4.00 \times 10^5 \, \text{N}) \times (2.50 \times 10^6 \, \text{m}) = 1.00 \times 10^{12} \, \text{J} \]

From (6), the final kinetic energy \( K_f = \frac{1}{2}mv_f^2 \) of the probe can be expressed in terms of the work \( W \) and the initial kinetic energy \( K_i = \frac{1}{2}mv_i^2 \) as

\[ K_f = W + K_i \]

Thus, from the known mass and initial speed we have

\[ K_f = (1.00 \times 10^{12} \, \text{J}) + \frac{1}{2}(5.00 \times 10^4 \, \text{kg})(1.10 \times 10^4 \, \text{m/s})^2 = 4.025 \times 10^{12} \, \text{J} \]

The final kinetic energy is \( K_f = \frac{1}{2}mv_f^2 \), so the final speed of the probe is

\[ v_f = \sqrt{\frac{2K_f}{m}} = \sqrt{\frac{2(4.025 \times 10^{12})}{5.00 \times 10^4}} \approx 1.27 \times 10^4 \, \text{m/s} \]
QUICK CHECK EXERCISES 6.6  (See page 458 for answers.)

1. If a constant force of 5 lb moves an object 10 ft, then the work done by the force on the object is ________.

2. A newton-meter is also called a ________. A dyne-centimeter is also called an ________.

3. Suppose that an object moves in the positive direction along a coordinate line over the interval \( [a, b] \). The work performed on the object by a variable force \( F(x) \) applied in the direction of motion is \( W = \) ________.

4. A force \( F(x) = 10 - 2x \) N applied in the positive \( x \)-direction moves an object 3 m from \( x = 2 \) to \( x = 5 \). The work done by the force on the object is ________.

EXERCISE SET 6.6

FOCUS ON CONCEPTS

1. A variable force \( F(x) \) in the positive \( x \)-direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from \( x = 0 \) to \( x = 3 \).

2. A variable force \( F(x) \) in the positive \( x \)-direction is graphed in the accompanying figure. Find the work done by the force on a particle that moves from \( x = 0 \) to \( x = 5 \).

3. For the variable force \( F(x) \) in Exercise 2, consider the distance \( d \) for which the work done by the force on the particle when the particle moves from \( x = 0 \) to \( x = d \) is half of the work done when the particle moves from \( x = 0 \) to \( x = 5 \). By inspecting the graph of \( F \), is \( d \) more or less than 2.5? Explain, and then find the exact value of \( d \).

4. Suppose that a variable force \( F(x) \) is applied in the positive \( x \)-direction so that an object moves from \( x = a \) to \( x = b \). Relate the work done by the force on the object and the average value of \( F \) over \( [a, b] \), and illustrate this relationship graphically.

5. A constant force of 10 lb in the positive \( x \)-direction is applied to a particle whose velocity versus time curve is shown in the accompanying figure. Find the work done by the force on the particle from time \( t = 0 \) to \( t = 5 \).

6. A spring exerts a force of 6 N when it is stretched from its natural length of 4 m to a length of \( 4 \frac{1}{2} \) m. Find the work required to stretch the spring from its natural length to a length of 6 m.

7. A spring exerts a force of 100 N when it is stretched 0.2 m beyond its natural length. How much work is required to stretch the spring 0.8 m beyond its natural length?

8. A spring whose natural length is 15 cm exerts a force of 45 N when stretched to a length of 20 cm.
   (a) Find the spring constant (in newtons/meter).
   (b) Find the work that is done in stretching the spring 3 cm beyond its natural length.
   (c) Find the work done in stretching the spring from a length of 20 cm to a length of 25 cm.

9. Assume that 10 ft·lb of work is required to stretch a spring 1 ft beyond its natural length. What is the spring constant?

10–13 True–False  Determine whether the statement is true or false. Explain your answer.

10. In order to support the weight of a parked automobile, the surface of a driveway must do work against the force of gravity on the vehicle.

11. A force of 10 lb in the direction of motion of an object that moves 5 ft in 2 s does six times the work of a force of 10 lb in the direction of motion of an object that moves 5 ft in 12 s.

12. It follows from Hooke’s law that in order to double the distance a spring is stretched beyond its natural length, four times as much work is required.

13. In the International System of Units, work and kinetic energy have the same units.
14. A cylindrical tank of radius 5 ft and height 9 ft is two-thirds filled with water. Find the work required to pump all the water over the upper rim.

15. Solve Exercise 14 assuming that the tank is half-filled with water.

16. A cone-shaped water reservoir is 20 ft in diameter across the top and 15 ft deep. If the reservoir is filled to a depth of 10 ft, how much work is required to pump all the water to the top of the reservoir?

17. The vat shown in the accompanying figure contains water to a depth of 2 m. Find the work required to pump all the water to the top of the vat. [Use 9810 N/m³ as the weight density of water.]

18. The cylindrical tank shown in the accompanying figure is filled with a liquid weighing 50 lb/ft³. Find the work required to pump all the liquid to a level 1 ft above the top of the tank.

19. A swimming pool is built in the shape of a rectangular parallelepiped 10 ft deep, 15 ft wide, and 20 ft long.

(a) If the pool is filled to 1 ft below the top, how much work is required to pump all the water into a drain at the top edge of the pool?

(b) A one-horsepower motor can do 550 ft-lb of work per second. What size motor is required to empty the pool in 1 hour?

20. How much work is required to fill the swimming pool in Exercise 19 to 1 ft below the top if the water is pumped in through an opening located at the bottom of the pool?

21. A 100 ft length of steel chain weighing 15 lb/ft is dangling from a pulley. How much work is required to wind the chain onto the pulley?

22. A 3 lb bucket containing 20 lb of water is hanging at the end of a 20 ft rope that weighs 4 oz/ft. The other end of the rope is attached to a pulley. How much work is required to wind the length of rope onto the pulley, assuming that the rope is wound onto the pulley at a rate of 2 ft/s and that as the bucket is being lifted, water leaks from the bucket at a rate of 0.5 lb/s?

23. A rocket weighing 3 tons is filled with 40 tons of liquid fuel. In the initial part of the flight, fuel is burned off at a constant rate of 2 tons per 1000 ft of vertical height. How much work in foot-tons (ft-ton) is done lifting the rocket 3000 ft?

24. It follows from Coulomb’s law in physics that two like electrostatic charges repel each other with a force inversely proportional to the square of the distance between them. Suppose that two charges $A$ and $B$ repel with a force of $k$ newtons when they are positioned at points $A(-a, 0)$ and $B(a, 0)$, where $a$ is measured in meters. Find the work $W$ required to move charge $A$ along the $x$-axis to the origin if charge $B$ remains stationary.

25. It follows from Newton’s Law of Universal Gravitation that the gravitational force exerted by the Earth on an object above the Earth’s surface varies inversely as the square of its distance from the Earth’s center. Thus, an object’s weight $w(x)$ is related to its distance $x$ from the Earth’s center by a formula of the form

$$w(x) = \frac{k}{x^2}$$

where $k$ is a constant of proportionality that depends on the mass of the object.

(a) Use this fact and the assumption that the Earth is a sphere of radius 4000 mi to obtain the formula for $w(x)$ in Example 4.

(b) Find a formula for the weight $w(x)$ of a satellite that is $x$ mi from the Earth’s surface if its weight on Earth is 6000 lb.

(c) How much work is required to lift the satellite from the surface of the Earth to an orbital position that is 1000 mi high?

26. (a) The formula $w(x) = k/x^2$ in Exercise 25 is applicable to all celestial bodies. Assuming that the Moon is a sphere of radius 1080 mi, find the force that the Moon exerts on an astronaut who is $x$ mi from the surface of the Moon if her weight on the Moon’s surface is 20 lb.

(b) How much work is required to lift the astronaut to a point that is 10.8 mi above the Moon’s surface?

27. The world’s first commercial high-speed magnetic levitation (MAGLEV) train, a 30 km double-track project connecting Shanghai, China, to Pudong International Airport, began full revenue service in 2003. Suppose that a MAGLEV train has a mass $m = 4.00 \times 10^4$ kg and that starting at a time when the train has a speed of 20 m/s the engine applies a force of $6.40 \times 10^5$ N in the direction of motion over a distance of 3.00 $\times 10^3$ m. Use the work–energy relationship (6) to find the final speed of the train.

28. Assume that a Mars probe of mass $m = 2.00 \times 10^3$ kg is subjected only to the force of its own engine. Starting at a time when the speed of the probe is $v = 1.00 \times 10^4$ m/s, the engine is fired continuously over a distance of 1.50 $\times 10^5$ m with a constant force of $2.00 \times 10^5$ N in the direction of motion. Use the work–energy relationship (6) to find the final speed of the probe.

29. On August 10, 1972 a meteorite with an estimated mass of $4 \times 10^6$ kg and an estimated speed of 15 km/s skipped across the atmosphere above the western United States and Canada but fortunately did not hit the Earth.

(Cont.)
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(a) Assuming that the meteorite had hit the Earth with a speed of 15 km/s, what would have been its change in kinetic energy in joules (J)?

(b) Express the energy as a multiple of the explosive energy of 1 megaton of TNT, which is $4.2 \times 10^{15}$ J.

(c) The energy associated with the Hiroshima atomic bomb was 13 kilotons of TNT. To how many such bombs would the meteorite impact have been equivalent?

30. Writing After reading Examples 3–5, a student classifies work problems as either “pushing/pulling” or “pumping.” Describe these categories in your own words and discuss the methods used to solve each type. Give examples to illustrate that these categories are not mutually exclusive.

31. Writing How might you recognize that a problem can be solved by means of the work–energy relationship? That is, what sort of “givens” and “unknowns” would suggest such a solution? Discuss two or three examples.

✓ QUICK CHECK ANSWERS 6.6

1. 50 ft·lb  2. joule; erg  3. $\int_a^b F(x) \, dx$  4. 9 J

6.7 MOMENTS, CENTERS OF GRAVITY, AND CENTROIDS

Suppose that a rigid physical body is acted on by a constant gravitational field. Because the body is composed of many particles, each of which is affected by gravity, the action of the gravitational field on the body consists of a large number of forces distributed over the entire body. However, it is a fact of physics that these individual forces can be replaced by a single force acting at a point called the center of gravity of the body. In this section we will show how integrals can be used to locate centers of gravity.

DENSITY AND MASS OF A LAMINA

Let us consider an idealized flat object that is thin enough to be viewed as a two-dimensional plane region (Figure 6.7.1). Such an object is called a lamina. A lamina is called homogeneous if its composition is uniform throughout and inhomogeneous otherwise. We will consider homogeneous laminas in this section. Inhomogeneous laminas will be discussed in Chapter 14. The density of a homogeneous lamina is defined to be its mass per unit area. Thus, the density $\delta$ of a homogeneous lamina of mass $M$ and area $A$ is given by $\delta = M / A$.

Notice that the mass $M$ of a homogeneous lamina can be expressed as

$$M = \delta A$$  

Example 1  A triangular lamina with vertices (0, 0), (0, 1), and (1, 0) has density $\delta = 3$. Find its total mass.

Solution. Referring to (1) and Figure 6.7.2, the mass $M$ of the lamina is

$$M = \delta A = 3 \cdot \frac{1}{2} = \frac{3}{2} \text{ (unit of mass)}$$

Center of Gravity of a Lamina

Assume that the acceleration due to the force of gravity is constant and acts downward, and suppose that a lamina occupies a region $R$ in a horizontal $xy$-plane. It can be shown that there exists a unique point $(\bar{x}, \bar{y})$ (which may or may not belong to $R$) such that the effect
of gravity on the lamina is “equivalent” to that of a single force acting at the point \((\bar{x}, \bar{y})\). This point is called the center of gravity of the lamina, and if it is in \(R\), then the lamina will balance horizontally on the point of a support placed at \((\bar{x}, \bar{y})\). For example, the center of gravity of a homogeneous disk is at the center of the disk, and the center of gravity of a homogeneous rectangular region is at the center of the rectangle. For an irregularly shaped homogeneous lamina, locating the center of gravity requires calculus.

### 6.7.1 Problem
Let \(f\) be a positive continuous function on the interval \([a, b]\). Suppose that a homogeneous lamina with constant density \(\delta\) occupies a region \(R\) in a horizontal \(xy\)-plane bounded by the graphs of \(y = f(x)\), \(y = 0\), \(x = a\), and \(x = b\). Find the coordinates \((\bar{x}, \bar{y})\) of the center of gravity of the lamina.

To motivate the solution, consider what happens if we try to balance the lamina on a knife-edge parallel to the \(x\)-axis. Suppose the lamina in Figure 6.7.3 is placed on a knife-edge along a line \(y = c\) that does not pass through the center of gravity. Because the lamina behaves as if its entire mass is concentrated at the center of gravity \((\bar{x}, \bar{y})\), the lamina will be rotationally unstable and the force of gravity will cause a rotation about \(y = c\). Similarly, the lamina will undergo a rotation if placed on a knife-edge along \(y = d\). However, if the knife-edge runs along the line \(y = \bar{y}\) through the center of gravity, the lamina will be in perfect balance. Similarly, the lamina will be in perfect balance on a knife-edge along the line \(x = \bar{x}\) through the center of gravity. This suggests that the center of gravity of a lamina can be determined as the intersection of two lines of balance, one parallel to the \(x\)-axis and the other parallel to the \(y\)-axis. In order to find these lines of balance, we will need some preliminary results about rotations.

---

### Figure 6.7.3

Children on a seesaw learn by experience that a lighter child can balance a heavier one by sitting farther from the fulcrum or pivot point. This is because the tendency for an object to produce rotation is proportional not only to its mass but also to the distance between the object and the fulcrum. To make this more precise, consider an \(x\)-axis, which we view as a weightless beam. If a mass \(m\) is located on the axis at \(x\), then the tendency for that mass to produce a rotation of the beam about a point \(a\) on the axis is measured by the following quantity, called the moment of \(m\) about \(x = a\):

\[
\text{moment of } m \text{ about } x = a = m(x - a)
\]
The number \( x - a \) is called the lever arm. Depending on whether the mass is to the right or left of \( a \), the lever arm is either the distance between \( x \) and \( a \) or the negative of this distance (Figure 6.7.4). Positive lever arms result in positive moments and clockwise rotations, and negative lever arms result in negative moments and counterclockwise rotations.

Suppose that masses \( m_1, m_2, \ldots, m_n \) are located at \( x_1, x_2, \ldots, x_n \) on a coordinate axis and a fulcrum is positioned at the point \( a \) (Figure 6.7.5). Depending on whether the sum of the moments about \( a \),

\[
\sum_{k=1}^{n} m_k (x_k - a) = m_1(x_1 - a) + m_2(x_2 - a) + \cdots + m_n(x_n - a)
\]

is positive, negative, or zero, a weightless beam along the axis will rotate clockwise about \( a \), rotate counterclockwise about \( a \), or balance perfectly. In the last case, the system of masses is said to be in equilibrium.

The preceding ideas can be extended to masses distributed in two-dimensional space. If we imagine the \( xy \)-plane to be a weightless sheet supporting a mass \( m \) located at a point \((x, y)\), then the tendency for the mass to produce a rotation of the sheet about the line \( x = a \) is \( m(x-a) \), called the moment of \( m \) about \( x = a \), and the tendency for the mass to produce a rotation about the line \( y = c \) is \( m(y-c) \), called the moment of \( m \) about \( y = c \) (Figure 6.7.6). In summary,

\[
\begin{align*}
\text{moment of } m \\
\text{about the line } x = a &= m(x-a) \\
\text{moment of } m \\
\text{about the line } y = c &= m(y-c)
\end{align*}
\]

(2–3)

If a number of masses are distributed throughout the \( xy \)-plane, then the plane (viewed as a weightless sheet) will balance on a knife-edge along the line \( x = a \) if the sum of the moments about that line are zero. Similarly, the plane will balance on a knife-edge along the line \( y = c \) if the sum of the moments about that line are zero.

We are now ready to solve Problem 6.7.1. The basic idea for solving this problem is to divide the lamina into strips whose areas may be approximated by the areas of rectangles. These area approximations, along with Formulas (2) and (3), will allow us to create a Riemann sum that approximates the moment of the lamina about a horizontal or vertical line. By taking the limit of Riemann sums we will then obtain an integral for the moment of a lamina about a horizontal or vertical line. We observe that since the lamina balances on the lines \( x = \bar{x} \) and \( y = \bar{y} \), the moment of the lamina about those lines should be zero. This observation will enable us to calculate \( \bar{x} \) and \( \bar{y} \).

To implement this idea, we divide the interval \([a, b]\) into \( n \) subintervals by inserting the points \( x_1, x_2, \ldots, x_n \) between \( a = x_0 \) and \( b = x_n \). This has the effect of dividing the lamina \( R \) into \( n \) strips \( R_1, R_2, \ldots, R_n \) (Figure 6.7.7a). Suppose that the \( k \)th strip extends from \( x_{k-1} \) to \( x_k \) and that the width of this strip is

\[
\Delta x_k = x_k - x_{k-1}
\]

We will let \( x_k^* \) be the midpoint of the \( k \)th subinterval and we will approximate \( R_k \) by a rectangle of width \( \Delta x_k \) and height \( f(x_k^*) \). From (1), the mass \( \Delta M_k \) of this rectangle is \( \Delta M_k = f(x_k^*) \Delta x_k \), and we will assume that the rectangle behaves as if its entire mass is concentrated at its center \((x_k^*, y_k^*) = (x_k^*, \frac{1}{2} f(x_k^*))\) (Figure 6.7.7b). It then follows from (2) and (3) that the moments of \( R_k \) about the lines \( x = \bar{x} \) and \( y = \bar{y} \) may be approximated
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by \((x_k^* - \bar{x})\Delta M_k\) and \((y_k^* - \bar{y})\Delta M_k\), respectively. Adding these approximations yields the following Riemann sums that approximate the moment of the entire lamina about the lines \(x = \bar{x}\) and \(y = \bar{y}\):

\[
\sum_{k=1}^{n}(x_k^* - \bar{x})\Delta M_k = \sum_{k=1}^{n}(x_k^* - \bar{x})\delta f(x_k^*)\Delta x_k
\]

\[
\sum_{k=1}^{n}(y_k^* - \bar{y})\Delta M_k = \sum_{k=1}^{n}\left(\frac{f(x_k^*)}{2} - \bar{y}\right)\delta f(x_k^*)\Delta x_k
\]

Taking the limits as \(n\) increases and the widths of all the rectangles approach zero yields the definite integrals

\[
\int_{a}^{b}(x - \bar{x})\delta f(x)dx
\]

\[
\int_{a}^{b}\left(\frac{f(x)}{2} - \bar{y}\right)\delta f(x)dx
\]

that represent the moments of the lamina about the lines \(x = \bar{x}\) and \(y = \bar{y}\). Since the lamina balances on those lines, the moments of the lamina about those lines should be zero:

\[
\int_{a}^{b}(x - \bar{x})\delta f(x)dx = \int_{a}^{b}\left(\frac{f(x)}{2} - \bar{y}\right)\delta f(x)dx = 0
\]

Since \(\bar{x}\) and \(\bar{y}\) are constant, these equations can be rewritten as

\[
\int_{a}^{b}\delta xf(x)dx = \bar{x}\int_{a}^{b}\delta f(x)dx
\]

\[
\int_{a}^{b}\frac{1}{2}\delta (f(x))^2 dx = \bar{y}\int_{a}^{b}\delta f(x)dx
\]

from which we obtain the following formulas for the center of gravity of the lamina:

\[
\text{Center of Gravity (}\bar{x}, \bar{y}\text{) of a Lamina}
\]

\[
\bar{x} = \frac{\int_{a}^{b}\delta xf(x)dx}{\int_{a}^{b}\delta f(x)dx}, \quad \bar{y} = \frac{\int_{a}^{b}\frac{1}{2}\delta (f(x))^2 dx}{\int_{a}^{b}\delta f(x)dx}
\] (4–5)

Observe that in both formulas the denominator is the mass \(M\) of the lamina. The numerator in the formula for \(\bar{x}\) is denoted by \(M_y\) and is called the first moment of the lamina about the \(y\)-axis; the numerator of the formula for \(\bar{y}\) is denoted by \(M_x\) and is called the first moment of the lamina about the \(x\)-axis. Thus, we can write (4) and (5) as

\[
\text{Alternative Formulas for Center of Gravity (}\bar{x}, \bar{y}\text{) of a Lamina}
\]

\[
\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R}\int_{a}^{b}\delta xf(x)dx
\]

\[
\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R}\int_{a}^{b}\frac{1}{2}\delta (f(x))^2 dx
\] (6)

\[
\text{Example 2} \quad \text{Find the center of gravity of the triangular lamina with vertices (0, 0), (0, 1), and (1, 0) and density } \delta = 3.
\]

\textbf{Solution.} \quad \text{The lamina is shown in Figure 6.7.2. In Example 1 we found the mass of the lamina to be}

\[
M = \frac{3}{2}
\]
The moment of the lamina about the \( y \)-axis is
\[
M_y = \int_0^1 \delta x f(x) \, dx = \int_0^1 3x(-x + 1) \, dx
\]
\[
= \int_0^1 (-3x^2 + 3x) \, dx = \left[-x^3 + \frac{3}{2} x^2\right]_0^1 = -1 + \frac{3}{2} = \frac{1}{2}
\]
and the moment about the \( x \)-axis is
\[
M_x = \int_0^1 \frac{1}{2} \delta (f(x))^2 \, dx = \int_0^1 \frac{1}{2} (-x + 1)^2 \, dx
\]
\[
= \int_0^1 \frac{3}{2} (x^2 - 2x + 1) \, dx = \frac{3}{2} \left(\frac{1}{3} x^3 - x^2 + x\right)\bigg|_0^1 = \frac{3}{2} \left(\frac{1}{3}\right) = \frac{1}{2}
\]
From (6) and (7),
\[
\bar{x} = \frac{M_y}{M} = \frac{1/2}{3/2} = \frac{1}{3}, \quad \bar{y} = \frac{M_x}{M} = \frac{1/2}{3/2} = \frac{1}{3}
\]
so the center of gravity is \((\frac{1}{3}, \frac{1}{3})\).

In the case of a homogeneous lamina, the center of gravity of a lamina occupying the region \( R \) is called the centroid of the region \( R \). Since the lamina is homogeneous, \( \delta \) is constant. The factor \( \delta \) in (4) and (5) may thus be moved through the integral signs and canceled, and (4) and (5) can be expressed as

\[
\text{Centroid of a Region } R
\]
\[
\bar{x} = \frac{1}{\text{area of } R} \int_a^b x f(x) \, dx \quad (8)
\]
\[
\bar{y} = \frac{1}{\text{area of } R} \int_a^b \frac{1}{2} (f(x))^2 \, dx \quad (9)
\]

**Example 3** Find the centroid of the semicircular region in Figure 6.7.8.

**Solution.** By symmetry, \( \bar{x} = 0 \) since the \( y \)-axis is obviously a line of balance. To find \( \bar{y} \), first note that the equation of the semicircle is \( y = f(x) = \sqrt{a^2 - x^2} \). From (9),
\[
\bar{y} = \frac{1}{\text{area of } R} \int_{-a}^a \frac{1}{2} (f(x))^2 \, dx = \frac{1}{\frac{1}{2} \pi a^2} \int_{-a}^a \frac{1}{2} (a^2 - x^2) \, dx
\]
\[
= \frac{1}{\pi a^2} \left[ a^3 x - \frac{1}{3} x^3 \right]_{-a}^a
\]
\[
= \frac{1}{\pi a^2} \left[ a^3 - \frac{1}{3} a^3 \right] - \left[-a^3 + \frac{1}{3} a^3\right]
\]
\[
= \frac{1}{\pi a^2} \left( \frac{4a^3}{3} \right) = \frac{4a}{3\pi}
\]
so the centroid is \((0, 4a/3\pi)\).
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OTHER TYPES OF REGIONS

The strategy used to find the center of gravity of the region in Problem 6.7.1 can be used to find the center of gravity of regions that are not of that form.

Consider a homogeneous lamina that occupies the region \( R \) between two continuous functions \( f(x) \) and \( g(x) \) over the interval \([a, b] \), where \( f(x) \geq g(x) \) for \( a \leq x \leq b \). To find the center of gravity of this lamina we can subdivide it into \( n \) strips using lines parallel to the \( y \)-axis. If \( x_k^* \) is the midpoint of the \( k \)th strip, the strip can be approximated by a rectangle of width \( \Delta x_k \) and height \( f(x_k^*) - g(x_k^*) \). We assume that the entire mass of the \( k \)th rectangle is concentrated at its center \((x_k^*, y_k^*) = \left( x_k^*, \frac{1}{2}(f(x_k^*) + g(x_k^*)) \right) \) (Figure 6.7.9). Continuing the argument as in the solution of Problem 6.7.1, we find that the center of gravity of the lamina is

\[
\bar{x} = \frac{\int_a^b x(f(x) - g(x)) \, dx}{\text{area of } R} = \frac{1}{\text{area of } R} \int_a^b x(f(x) - g(x)) \, dx \tag{10}
\]

\[
\bar{y} = \frac{\int_a^b \frac{1}{2} \left( [f(x)]^2 - [g(x)]^2 \right) \, dx}{\text{area of } R} = \frac{1}{\text{area of } R} \int_a^b \frac{1}{2} \left( [f(x)]^2 - [g(x)]^2 \right) \, dx \tag{11}
\]

Note that the density of the lamina does not appear in Equations (10) and (11). This reflects the fact that the centroid is a geometric property of \( R \).

Example 4

Find the centroid of the region \( R \) enclosed between the curves \( y = x^2 \) and \( y = x + 6 \).

Solution. To begin, we note that the two curves intersect when \( x = -2 \) and \( x = 3 \) and that \( x + 6 \geq x^2 \) over that interval (Figure 6.7.10). The area of \( R \) is

\[
\int_{-2}^{3} [(x + 6) - x^2] \, dx = \frac{125}{6}
\]

From (10) and (11),

\[
\bar{x} = \frac{1}{\text{area of } R} \int_{-2}^{3} x[(x + 6) - x^2] \, dx = \frac{6}{125} \left[ \frac{1}{3} x^3 + 3x^2 - \frac{1}{4} x^4 \right]_{-2}^{3} = 6 \cdot \frac{125}{12} = \frac{1}{2}
\]

and

\[
\bar{y} = \frac{1}{\text{area of } R} \int_{-2}^{3} \frac{1}{2} ((x + 6)^2 - (x^2)^2) \, dx = \frac{6}{125} \int_{-2}^{3} \frac{1}{2} (x^2 + 12x + 36 - x^4) \, dx = \frac{6}{125} \cdot \frac{250}{3} = 4
\]

so the centroid of \( R \) is \( \left( \frac{1}{2}, 4 \right) \).
Suppose that \( w \) is a continuous function of \( y \) on an interval \([c, d]\) with \( w(y) \geq 0 \) for \( c \leq y \leq d \). Consider a lamina that occupies a region \( R \) bounded above by \( y = d \), below by \( y = c \), on the left by the \( y \)-axis, and on the right by \( x = w(y) \) (Figure 6.7.11). To find the center of gravity of this lamina, we note that the roles of \( x \) and \( y \) in Problem 6.7.1 have been reversed. We now imagine the lamina to be subdivided into \( n \) strips using lines parallel to the \( x \)-axis. We let \( y^*_k \) be the midpoint of the \( k \)th subinterval and approximate the strip by a rectangle of width \( \Delta_1 y_k \) and height \( w(y^*_k) \). We assume that the entire mass of the \( k \)th rectangle is concentrated at its center \((x^*_k, y^*_k) = \left( \frac{1}{2} w(y^*_k), y^*_k \right) \) (Figure 6.7.11). Continuing the argument as in the solution of Problem 6.7.1, we find that the center of gravity of the lamina is

\[
\bar{x} = \frac{\int_c^d \frac{1}{2} (w(y))^2 \, dy}{\int_c^d w(y) \, dy} = \frac{1}{\text{area of } R} \int_c^d \frac{1}{2} (w(y))^2 \, dy \tag{12}
\]

\[
\bar{y} = \frac{\int_c^d y w(y) \, dy}{\int_c^d w(y) \, dy} = \frac{1}{\text{area of } R} \int_c^d y w(y) \, dy \tag{13}
\]

Once again, the absence of the density in Equations (12) and (13) reflects the geometric nature of the centroid.

**Example 5** Find the centroid of the region \( R \) enclosed between the curves \( y = \sqrt{x} \), \( y = 1 \), \( y = 2 \), and the \( y \)-axis (Figure 6.7.12).

**Solution.** Note that \( x = w(y) = y^2 \) and that the area of \( R \) is

\[
\int_1^2 y^2 \, dy = \frac{7}{3}
\]

From (12) and (13),

\[
\bar{x} = \frac{1}{\text{area of } R} \int_1^2 \frac{1}{2} (y^2)^2 \, dy = \frac{1}{7} \cdot \frac{1}{10} y^5 \bigg|_1^2 = \frac{3}{7} \cdot \frac{31}{10} = \frac{93}{70}
\]

\[
\bar{y} = \frac{1}{\text{area of } R} \int_1^2 y(y^2) \, dy = \frac{3}{7} \cdot \frac{1}{4} y^4 \bigg|_1^2 = \frac{3}{7} \cdot \frac{15}{4} = \frac{45}{28}
\]

so the centroid of \( R \) is \((93/70, 45/28) \approx (1.329, 1.607)\).

**THEOREM OF PAPPUS**

The following theorem, due to the Greek mathematician Pappus, gives an important relationship between the centroid of a plane region \( R \) and the volume of the solid generated when the region is revolved about a line.

**6.7.2 Theorem (Theorem of Pappus)** If \( R \) is a bounded plane region and \( L \) is a line that lies in the plane of \( R \) such that \( R \) is entirely on one side of \( L \), then the volume of the solid formed by revolving \( R \) about \( L \) is given by

\[
\text{volume} = (\text{area of } R) \cdot \left( \text{distance traveled by the centroid} \right)
\]
### 6.7 Moments, Centers of Gravity, and Centroids

**Proof** We prove this theorem in the special case where \( L \) is the \( y \)-axis, the region \( R \) is in the first quadrant, and the region \( R \) is of the form given in Problem 6.7.1. (A more general proof will be outlined in the Exercises of Section 14.8.) In this case, the volume \( V \) of the solid formed by revolving \( R \) about \( L \) can be found by the method of cylindrical shells (Section 6.3) to be

\[
V = 2\pi \int_a^b xf(x) \, dx
\]

Thus, it follows from (8) that

\[
V = 2\pi \bar{x} \text{[area of } R]\]

This completes the proof since \( 2\pi \bar{x} \) is the distance traveled by the centroid when \( R \) is revolved about the \( y \)-axis. \( \blacksquare \)

**Example 6** Use Pappus’ Theorem to find the volume \( V \) of the torus generated by revolving a circular region of radius \( b \) about a line at a distance \( a \) (greater than \( b \)) from the center of the circle (Figure 6.7.13).

**Solution.** By symmetry, the centroid of a circular region is its center. Thus, the distance traveled by the centroid is \( 2\pi a \). Since the area of a circle of radius \( b \) is \( \pi b^2 \), it follows from Pappus’ Theorem that the volume of the torus is

\[
V = (2\pi a)(\pi b^2) = 2\pi^2 ab^2
\]

✔

**Quick Check Exercises 6.7** (See page 467 for answers.)

1. The total mass of a homogeneous lamina of area \( A \) and density \( \delta \) is ________.

2. A homogeneous lamina of mass \( M \) and density \( \delta \) occupies a region in the \( xy \)-plane bounded by the graphs of \( y = f(x) \), \( y = 0 \), \( x = a \), and \( x = b \), where \( f \) is a nonnegative continuous function defined on an interval \([a, b]\). The \( x \)-coordinate of the center of gravity of the lamina is \( M_y/M \), where \( M_y \) is called the ________ and is given by the integral ________.

3. Let \( R \) be the region between the graphs of \( y = x^2 \) and \( y = 2 - x \) for \( 0 \leq x \leq 1 \). The area of \( R \) is \( \frac{7}{6} \) and the centroid of \( R \) is ________.

4. If the region \( R \) in Quick Check Exercise 3 is used to generate a solid \( G \) by rotating \( R \) about a horizontal line 6 units above its centroid, then the volume of \( G \) is ________.

**Exercise Set 6.7**

**Focus on Concepts**

1. Masses \( m_1 = 5 \), \( m_2 = 10 \), and \( m_3 = 20 \) are positioned on a weightless beam as shown in the accompanying figure.

(a) Suppose that the fulcrum is positioned at \( x = 5 \). Without computing the sum of moments about 5, determine whether the sum is positive, zero, or negative. Explain.

(b) Where should the fulcrum be placed so that the beam is in equilibrium?

**Figure Ex-1**

---

**Pappus of Alexandria** (4th century A.D.) Greek mathematician. Pappus lived during the early Christian era when mathematical activity was in a period of decline. His main contributions to mathematics appeared in a series of eight books called The Collection (written about 340 A.D.). This work, which survives only partially, contained some original results but was devoted mostly to statements, refinements, and proofs of results by earlier mathematicians. Pappus’ Theorem, stated without proof in Book VII of The Collection, was probably known and proved in earlier times. This result is sometimes called Guldin’s Theorem in recognition of the Swiss mathematician, Paul Guldin (1577–1643), who rediscovered it independently.
2. Masses \( m_1 = 10, m_2 = 3, m_3 = 4, \) and \( m \) are positioned on a weightless beam, with the fulcrum positioned at point 4, as shown in the accompanying figure.
   (a) Suppose that \( m = 14 \). Without computing the sum of the moments about 4, determine whether the sum is positive, zero, or negative. Explain.
   (b) For what value of \( m \) is the beam in equilibrium?

3–6 Find the centroid of the region by inspection and confirm your answer by integrating.

3.

4.

5.

6.

7–20 Find the centroid of the region.

7.

8.

9.

10.

11. The triangle with vertices \((0, 0), (2, 0), \) and \((0, 1)\).

12. The triangle with vertices \((0, 0), (1, 1), \) and \((2, 0)\).

13. The region bounded by the graphs of \( y = x^2 \) and \( x + y = 6 \).

14. The region bounded on the left by the \( y \)-axis, on the right by the line \( x = 2 \), below by the parabola \( y = x^2 \), and above by the line \( y = x + 6 \).

15. The region bounded by the graphs of \( y = x^2 \) and \( y = x + 2 \).

16. The region bounded by the graphs of \( y = x^3 \) and \( y = 1 \).

17. The region bounded by the graphs of \( y = \sqrt{x} \) and \( y = x \).

18. The region bounded by the graphs of \( x = 1/y, x = 0, y = 1, \) and \( y = 2 \).

19. The region bounded by the graphs of \( y = x, x = 1/y^2, \) and \( y = 2 \).

20. The region bounded by the graphs of \( xy = 4 \) and \( x + y = 5 \).

21. Use symmetry considerations to argue that the centroid of an isosceles triangle lies on the median to the base of the triangle.

22. Use symmetry considerations to argue that the centroid of an ellipse lies at the intersection of the major and minor axes of the ellipse.

23–26 Find the mass and center of gravity of the lamina with density \( \delta \).

23. A lamina bounded by the \( x \)-axis, the line \( x = 1 \), and the curve \( y = \sqrt{x}; \delta = 2 \).

24. A lamina bounded by the graph of \( x = y^4 \) and the line \( x = 1 \); \( \delta = 15 \).

25. A lamina bounded by the graph of \( y = |x| \) and the line \( y = 1; \delta = 3 \).

26. A lamina bounded by the \( x \)-axis and the graph of the equation \( y = 1 - x^2; \delta = 3 \).

27–30 Use a CAS to find the mass and center of gravity of the lamina with density \( \delta \).

27. A lamina bounded by \( y = \sin x, y = 0, x = 0, \) and \( x = \pi; \delta = 4 \).

28. A lamina bounded by \( y = e^x, y = 0, x = 0, \) and \( x = 1; \delta = 1/(e - 1) \).

29. A lamina bounded by the graph of \( y = \ln x \), the \( x \)-axis, and the line \( x = 2; \delta = 1 \).

30. A lamina bounded by the graphs of \( y = \cos x, y = \sin x, x = 0, \) and \( x = \pi/4; \delta = 1 + \sqrt{2} \).

31–34 True–False Determine whether the statement is true or false. Explain your answer. [In Exercise 34, assume that the (rotated) square lies in the \( xy \)-plane to the right of the \( y \)-axis.]
31. The centroid of a rectangle is the intersection of the diagonals of the rectangle.
32. The centroid of a rhombus is the intersection of the diagonals of the rhombus.
33. The centroid of an equilateral triangle is the intersection of the medians of the triangle.
34. By rotating a square about its center, it is possible to change the volume of the solid of revolution generated by revolving the square about the y-axis.
35. Find the centroid of the triangle with vertices \((0, 0), (a, b), (a, -b)\).
36. Prove that the centroid of a triangle is the point of intersection of the medians of the triangle. [Hint: Choose coordinates so that the vertices of the triangle are located at \((0, -a), (0, a), (b, c)\).]
37. Find the centroid of the isosceles trapezoid with vertices \((-a, 0), (a, 0), (-b, c), (b, c)\).
38. Prove that the centroid of a parallelogram is the point of intersection of the diagonals of the parallelogram. [Hint: Choose coordinates so that the vertices of the parallelogram are located at \((0, 0), (0, a), (b, c), (b, a + c)\).]
39. Use the Theorem of Pappus and the fact that the volume of a sphere of radius \(a\) is \(V = \frac{4}{3}\pi a^3\) to show that the centroid of the lamina that is bounded by the \(x\)-axis and the semicircle \(y = \sqrt{a^2 - x^2}\) is \((0, 4a/(3\pi))\). (This problem was solved directly in Example 3.)
40. Use the Theorem of Pappus and the result of Exercise 39 to find the volume of the solid generated when the region bounded by the \(x\)-axis and the semicircle \(y = \sqrt{a^2 - x^2}\) is revolved about (a) the line \(y = -a\) (b) the line \(y = x - a\).
41. Use the Theorem of Pappus and the fact that the area of an ellipse with semiaxes \(a\) and \(b\) is \(\pi ab\) to find the volume of the elliptical torus generated by revolving the ellipse \(\frac{(x - k)^2}{a^2} + \frac{y^2}{b^2} = 1\) about the \(y\)-axis. Assume that \(k > a\).
42. Use the Theorem of Pappus to find the volume of the solid that is generated when the region enclosed by \(y = x^2\) and \(y = 8 - x^2\) is revolved about the \(x\)-axis.
43. Use the Theorem of Pappus to find the centroid of the triangular region with vertices \((0, 0), (a, 0), (0, b)\), where \(a > 0\) and \(b > 0\). [Hint: Revolve the region about the \(x\)-axis to obtain \(\bar{y}\) and about the \(y\)-axis to obtain \(\bar{x}\).]
44. Writing Suppose that a region \(R\) in the plane is decomposed into two regions \(R_1\) and \(R_2\) whose areas are \(A_1\) and \(A_2\), respectively, and whose centroids are \((\bar{x}_1, \bar{y}_1)\) and \((\bar{x}_2, \bar{y}_2)\), respectively. Investigate the problem of expressing the centroid of \(R\) in terms of \(A_1, A_2, (\bar{x}_1, \bar{y}_1),\) and \((\bar{x}_2, \bar{y}_2)\). Write a short report on your investigations, supporting your reasoning with plausible arguments. Can you extend your results to decompositions of \(R\) into more than two regions?
45. Writing How might you recognize that a problem can be solved by means of the Theorem of Pappus? That is, what sort of “givens” and “unknowns” would suggest such a solution? Discuss two or three examples.

### QUICK CHECK ANSWERS 6.7

1. \(\delta A\)  
2. first moment about the y-axis; \(\int_a^b \delta x f(x) \, dx\)  
3. \(\left(\frac{5}{14}, \frac{32}{35}\right)\)  
4. \(14\pi\)

### 6.8 FLUID PRESSURE AND FORCE

In this section we will use the integration tools developed in the preceding chapter to study the pressures and forces exerted by fluids on submerged objects.

#### WHAT IS A FLUID?

A fluid is a substance that flows to conform to the boundaries of any container in which it is placed. Fluids include liquids, such as water, oil, and mercury, as well as gases, such as helium, oxygen, and air. The study of fluids falls into two categories: fluid statics (the study of fluids at rest) and fluid dynamics (the study of fluids in motion). In this section we will be concerned only with fluid statics; toward the end of this text we will investigate problems in fluid dynamics.
THE CONCEPT OF PRESSURE

The effect that a force has on an object depends on how that force is spread over the surface of the object. For example, when you walk on soft snow with boots, the weight of your body crushes the snow and you sink into it. However, if you put on a pair of snowshoes to spread the weight of your body over a greater surface area, then the weight of your body has less of a crushing effect on the snow. The concept that accounts for both the magnitude of a force and the area over which it is applied is called pressure.

6.8.1 DEFINITION If a force of magnitude $F$ is applied to a surface of area $A$, then we define the pressure $P$ exerted by the force on the surface to be

$$P = \frac{F}{A}$$

It follows from this definition that pressure has units of force per unit area. The most common units of pressure are newtons per square meter (N/m$^2$) in SI and pounds per square inch (lb/in$^2$) or pounds per square foot (lb/ft$^2$) in the BE system. As indicated in Table 6.8.1, one newton per square meter is called a pascal (Pa). A pressure of 1 Pa is quite small (1 Pa = $1.45 \times 10^{-4}$ lb/in$^2$), so in countries using SI, tire pressure gauges are usually calibrated in kilopascals (kPa), which is 1000 pascals.

<table>
<thead>
<tr>
<th>SYSTEM</th>
<th>FORCE</th>
<th>AREA</th>
<th>PRESSURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>SI</td>
<td>newton (N)</td>
<td>square meter (m$^2$)</td>
<td>pascal (Pa)</td>
</tr>
<tr>
<td>BE</td>
<td>pound (lb)</td>
<td>square foot (ft$^2$)</td>
<td>lb/ft$^2$</td>
</tr>
<tr>
<td>BE</td>
<td>pound (lb)</td>
<td>square inch (in$^2$)</td>
<td>lb/in$^2$ (psi)</td>
</tr>
</tbody>
</table>

**CONVERSION FACTORS:**

- 1 Pa = $1.45 \times 10^{-4}$ lb/in$^2$ = $2.09 \times 10^{-2}$ lb/ft$^2$
- 1 lb/in$^2$ = $6.89 \times 10^3$ Pa
- 1 lb/ft$^2$ = 47.9 Pa

**Blaise Pascal** (1623–1662) French mathematician and scientist. Pascal’s mother died when he was three years old and his father, a highly educated magistrate, personally provided the boy's early education. Although Pascal showed an inclination for science and mathematics, his father refused to tutor him in those subjects until he mastered Latin and Greek. Pascal’s sister and primary biographer claimed that he independently discovered the first thirty-two propositions of Euclid without ever reading a book on geometry. (However, it is generally agreed that the story is apocryphal.) Nevertheless, the precocious Pascal published a highly respected essay on conic sections by the time he was sixteen years old. Descartes, who read the essay, thought it so brilliant that he could not believe that it was written by such a young man. By age 18 his health began to fail and until his death he was in frequent pain. However, his creativity was unimpaired.

Pascal’s contributions to physics include the discovery that air pressure decreases with altitude and the principle of fluid pressure that bears his name. However, the originality of his work is questioned by some historians. Pascal made major contributions to a branch of mathematics called “projective geometry,” and he helped to develop probability theory through a series of letters with Fermat.

In 1646, Pascal’s health problems resulted in a deep emotional crisis that led him to become increasingly concerned with religious matters. Although born a Catholic, he converted to a religious doctrine called Jansenism and spent most of his final years writing on religion and philosophy.
Fluid forces always act perpendicular to the surface of a submerged object.

### Example 1
Referring to Figure 6.8.1, suppose that the back of the swimmer’s hand has a surface area of \(8.4 \times 10^{-3} \text{ m}^2\) and that the pressure acting on it is \(5.1 \times 10^4 \text{ Pa}\) (a realistic value near the bottom of a deep diving pool). Find the force that acts on the swimmer’s hand.

**Solution.** From (1), the force \(F\) is
\[
F = PA = (5.1 \times 10^4 \text{ N/m}^2)(8.4 \times 10^{-3} \text{ m}^2) \approx 4.3 \times 10^2 \text{ N}
\]
This is quite a large force (nearly 100 lb in the BE system).

### Fluid Density
Scuba divers know that the pressure and forces on their bodies increase with the depth they dive. This is caused by the weight of the water and air above—the deeper the diver goes, the greater the weight above and so the greater the pressure and force exerted on the diver.

To calculate pressures and forces on submerged objects, we need to know something about the characteristics of the fluids in which they are submerged. For simplicity, we will assume that the fluids under consideration are *homogeneous*, by which we mean that any two samples of the fluid with the same volume have the same mass. It follows from this assumption that the mass per unit volume is a constant \(\delta\) that depends on the physical characteristics of the fluid but not on the size or location of the sample; we call
\[
\delta = \frac{m}{V} \quad (2)
\]
the **mass density** of the fluid. Sometimes it is more convenient to work with weight per unit volume rather than mass per unit volume. Thus, we define the **weight density** \(\rho\) of a fluid to be
\[
\rho = \frac{w}{V} \quad (3)
\]
where \(w\) is the weight of a fluid sample of volume \(V\). Thus, if the weight density of a fluid is known, then the weight \(w\) of a fluid sample of volume \(V\) can be computed from the formula \(w = \rho V\). Table 6.8.2 shows some typical weight densities.

### Fluid Pressure
To calculate fluid pressures and forces we will need to make use of an experimental observation. Suppose that a flat surface of area \(A\) is submerged in a homogeneous fluid of weight density \(\rho\) such that the entire surface lies between depths \(h_1\) and \(h_2\), where \(h_1 \leq h_2\) (Figure 6.8.2). Experiments show that on both sides of the surface, the fluid exerts a force that is perpendicular to the surface and whose magnitude \(F\) satisfies the inequalities
\[
\rho h_1 A \leq F \leq \rho h_2 A \quad (4)
\]
Thus, it follows from (1) that the pressure \(P = F/A\) on a given side of the surface satisfies the inequalities
\[
\rho h_1 \leq P \leq \rho h_2 \quad (5)
\]
Note that it is now a straightforward matter to calculate fluid force and pressure on a flat surface that is submerged horizontally at depth \( h \), for then \( h = h_1 = h_2 \) and inequalities (4) and (5) become the equalities
\[
F = \rho h A \quad \text{(6)}
\]
and
\[
P = \rho h \quad \text{(7)}
\]

**Example 2** Find the fluid pressure and force on the top of a flat circular plate of radius \( 2 \) m that is submerged horizontally in water at a depth of \( 6 \) m (Figure 6.8.3).

**Solution.** Since the weight density of water is \( \rho = 9810 \) N/m\(^3\), it follows from (7) that the fluid pressure is
\[
P = \rho h = (9810)(6) = 58,860 \text{ Pa}
\]
and it follows from (6) that the fluid force is
\[
F = \rho h A = \rho h(\pi r^2) = (9810)(6)(4\pi) = 235,440\pi \approx 739,700 \text{ N}
\]

### FLUID FORCE ON A VERTICAL SURFACE

It was easy to calculate the fluid force on the horizontal plate in Example 2 because each point on the plate was at the same depth. The problem of finding the fluid force on a vertical surface is more complicated because the depth, and hence the pressure, is not constant over the surface. To find the fluid force on a vertical surface we will need calculus.

6.8.2 **Problem** Suppose that a flat surface is immersed vertically in a fluid of weight density \( \rho \) and that the submerged portion of the surface extends from \( x = a \) to \( x = b \) along an \( x \)-axis whose positive direction is down (Figure 6.8.4a). For \( a \leq x \leq b \), suppose that \( w(x) \) is the width of the surface and that \( h(x) \) is the depth of the point \( x \). Define what is meant by the fluid force \( F \) on the surface, and find a formula for computing it.

The basic idea for solving this problem is to divide the surface into horizontal strips whose areas may be approximated by areas of rectangles. These area approximations, along with inequalities (4), will allow us to create a Riemann sum that approximates the total force on the surface. By taking a limit of Riemann sums we will then obtain an integral for \( F \).

To implement this idea, we divide the interval \([a, b]\) into \( n \) subintervals by inserting the points \( x_1, x_2, \ldots, x_{n-1} \) between \( a = x_0 \) and \( b = x_n \). This has the effect of dividing the surface into \( n \) strips of area \( A_k, k = 1, 2, \ldots, n \) (Figure 6.8.4b). It follows from (4) that the force \( F_k \) on the \( k \)th strip satisfies the inequalities
\[
\rho h(x_{k-1})A_k \leq F_k \leq \rho h(x_k)A_k
\]
or, equivalently,
\[
h(x_{k-1}) \leq \frac{F_k}{\rho A_k} \leq h(x_k)
\]
Since the depth function \( h(x) \) increases linearly, there must exist a point \( x_k^* \) between \( x_{k-1} \) and \( x_k \) such that
\[
h(x_k^*) = \frac{F_k}{\rho A_k}
\]
or, equivalently,
\[
F_k = \rho h(x_k^*)A_k
\]
6.8 Fluid Pressure and Force

We now approximate the area $A_k$ of the $k$th strip of the surface by the area of a rectangle of width $w(x_k^*)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.8.4c). It follows that $F_k$ may be approximated as

$$F_k = \rho h(x_k^*) A_k \approx \rho h(x_k^*) \frac{w(x_k^*) \Delta x_k}{\text{Area of rectangle}}$$

Adding these approximations yields the following Riemann sum that approximates the total force $F$ on the surface:

$$F = \sum_{k=1}^{n} F_k \approx \sum_{k=1}^{n} \rho h(x_k^*) w(x_k^*) \Delta x_k$$

Taking the limit as $n$ increases and the widths of all the subintervals approach zero yields the definite integral

$$F = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \rho h(x_k^*) w(x_k^*) \Delta x_k = \int_{a}^{b} \rho h(x) w(x) \, dx$$

In summary, we have the following result.

6.8.3 Definition Suppose that a flat surface is immersed vertically in a fluid of weight density $\rho$ and that the submerged portion of the surface extends from $x = a$ to $x = b$ along an $x$-axis whose positive direction is down (Figure 6.8.4a). For $a \leq x \leq b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point $x$. Then we define the fluid force $F$ on the surface to be

$$F = \int_{a}^{b} \rho h(x) w(x) \, dx$$

(8)

Example 3 The face of a dam is a vertical rectangle of height 100 ft and width 200 ft (Figure 6.8.5a). Find the total fluid force exerted on the face when the water surface is level with the top of the dam.

Solution. Introduce an $x$-axis with its origin at the water surface as shown in Figure 6.8.5b. At a point $x$ on this axis, the width of the dam in feet is $w(x) = 200$ and the depth in feet is $h(x) = x$. Thus, from (8) with $\rho = 62.4 \text{ lb/ft}^3$ (the weight density of water) we obtain as the total force on the face

$$F = \int_{0}^{100} (62.4)(x)(200) \, dx = 12,480 \int_{0}^{100} x \, dx$$

$$= 12,480 \left[ \frac{x^2}{2} \right]_{0}^{100} = 62,400,000 \text{ lb}$$

Example 4 A plate in the form of an isosceles triangle with base 10 ft and altitude 4 ft is submerged vertically in machine oil as shown in Figure 6.8.6a. Find the fluid force $F$ against the plate surface if the oil has weight density $\rho = 30 \text{ lb/ft}^3$.

Solution. Introduce an $x$-axis as shown in Figure 6.8.6b. By similar triangles, the width of the plate, in feet, at a depth of $h(x) = (3 + x)$ ft satisfies

$$\frac{w(x)}{10} = \frac{x}{4}, \quad \text{so} \quad w(x) = \frac{5}{2} x$$
Thus, it follows from (8) that the force on the plate is
\[
F = \int_a^b \rho h(x)w(x) \, dx = \int_0^4 (30)(3 + x) \left( \frac{5}{2}x \right) \, dx \\
= 75 \int_0^4 (3x + x^2) \, dx = 75 \left[ \frac{3x^2}{2} + \frac{x^3}{3} \right]_0^4 = 3400 \text{ lb}
\]

1. The pressure unit equivalent to a newton per square meter (N/m²) is called a _______. The pressure unit psi stands for _______.
2. Given that the weight density of water is 9810 N/m³, the fluid pressure on a rectangular 2 m × 3 m flat plate submerged horizontally in water at a depth of 10 m is _______. The fluid force on the plate is _______.
3. Suppose that a flat surface is immersed vertically in a fluid of weight density ρ and that the submerged portion of the surface extends from x = a to x = b along an x-axis whose positive direction is down. If, for a ≤ x ≤ b, the surface has width w(x) and depth h(x), then the fluid force on the surface is F = _______.
4. A rectangular plate 2 m wide and 3 m high is submerged vertically in water so that the top of the plate is 5 m below the water surface. An integral expression for the force of the water on the plate surface is F = _______.

EXERCISE SET 6.8

In this exercise set, refer to Table 6.8.2 for weight densities of fluids, where needed.

1. A flat rectangular plate is submerged horizontally in water.
   (a) Find the force (in lb) and the pressure (in lb/ft²) on the top surface of the plate if its area is 100 ft² and the surface is at a depth of 5 ft.
   (b) Find the force (in N) and the pressure (in Pa) on the top surface of the plate if its area is 25 m² and the surface is at a depth of 10 m.

2. (a) Find the force (in N) on the deck of a sunken ship if its area is 160 m² and the pressure acting on it is 6.0 × 10⁵ Pa.
   (b) Find the force (in lb) on a diver’s face mask if its area is 60 in² and the pressure acting on it is 100 lb/in².

3–8 The flat surfaces shown are submerged vertically in water. Find the fluid force against each surface.

9. Suppose that a flat surface is immersed vertically in a fluid of weight density ρ. If ρ is doubled, is the force on the plate also doubled? Explain your reasoning.

10. An oil tank is shaped like a right circular cylinder of diameter 4 ft. Find the total fluid force against one end when the axis is horizontal and the tank is half filled with oil of weight density 50 lb/ft³.

11. A square plate of side a feet is dipped in a liquid of weight density ρ lb/ft³. Find the fluid force on the plate if a vertex is at the surface and a diagonal is perpendicular to the surface.

12–15 True–False Determine whether the statement is true or false. Explain your answer.

12. In the International System of Units, pressure and force have the same units.
13. In a cylindrical water tank (with vertical axis), the fluid force on the base of the tank is equal to the weight of water in the tank.
14. In a rectangular water tank, the fluid force on any side of the tank must be less than the fluid force on the base of the tank.
6.8 Fluid Pressure and Force

15. In any water tank with a flat base, no matter what the shape of the tank, the fluid force on the base is at most equal to the weight of water in the tank.

16–19 Formula (8) gives the fluid force on a flat surface immersed vertically in a fluid. More generally, if a flat surface is immersed so that it makes an angle of $0 \leq \theta < \pi / 2$ with the vertical, then the fluid force on the surface is given by

$$F = \int_a^b \rho h(x)w(x) \sec \theta \, dx$$

Use this formula in these exercises.

16. Derive the formula given above for the fluid force on a flat surface immersed at an angle in a fluid.

17. The accompanying figure shows a rectangular swimming pool whose bottom is an inclined plane. Find the fluid force on the bottom when the pool is filled to the top.

18. By how many feet should the water in the pool of Exercise 17 be lowered in order for the force on the bottom to be reduced by a factor of $1/2$?

19. The accompanying figure shows a dam whose face is an inclined rectangle. Find the fluid force on the face when the water is level with the top of this dam.

20. An observation window on a submarine is a square with 2 ft sides. Using $\rho_0$ for the weight density of seawater, find the fluid force on the window when the submarine has descended so that the window is vertical and its top is at a depth of $h$ feet.

FOCUS ON CONCEPTS

21. (a) Show: If the submarine in Exercise 20 descends vertically at a constant rate, then the fluid force on the window increases at a constant rate.

(b) At what rate is the force on the window increasing if the submarine is descending vertically at 20 ft/min?

22. (a) Let $D = D_a$ denote a disk of radius $a$ submerged in a fluid of weight density $\rho$ such that the center of $D$ is $h$ units below the surface of the fluid. For each value of $r$ in the interval $[0, a]$, let $D_r$ denote the disk of radius $r$ that is concentric with $D$. Select a side of the disk $D$ and define $P(r)$ to be the fluid pressure on the chosen side of $D_r$. Use (5) to prove that

$$\lim_{r \to 0^+} P(r) = \rho h$$

(b) Explain why the result in part (a) may be interpreted to mean that fluid pressure at a given depth is the same in all directions. (This statement is one version of a result known as Pascal’s Principle.)

23. Writing Suppose that we model the Earth’s atmosphere as a “fluid.” Atmospheric pressure at sea level is $P = 14.7$ lb/in² and the weight density of air at sea level is about $\rho = 4.66 \times 10^{-5}$ lb/in³. With these numbers, what would Formula (7) yield as the height of the atmosphere above the Earth? Do you think this answer is reasonable? If not, explain how we might modify our assumptions to yield a more plausible answer.

24. Writing Suppose that the weight density $\rho$ of a fluid is a function $\rho = \rho(x)$ of the depth $x$ within the fluid. How do you think that Formula (7) for fluid pressure will need to be modified? Support your answer with plausible arguments.

✔ QUICK CHECK ANSWERS 6.8

1. pascal; pounds per square inch
2. 98,100 Pa; 588,600 N
3. $\int_a^b \rho h(x)w(x) \, dx$
4. $\int_0^3 9810 [(5 + x)2] \, dx$
In this section we will study certain combinations of \(e^x\) and \(e^{-x}\), called “hyperbolic functions.” These functions, which arise in various engineering applications, have many properties in common with the trigonometric functions. This similarity is somewhat surprising, since there is little on the surface to suggest that there should be any relationship between exponential and trigonometric functions. This is because the relationship occurs within the context of complex numbers, a topic which we will leave for more advanced courses.

### Definitions of Hyperbolic Functions

To introduce the hyperbolic functions, observe from Exercise 65 in Section 0.2 that the function \(e^x\) can be expressed in the following way as the sum of an even function and an odd function:

\[
e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}
\]

These functions are sufficiently important that there are names and notation associated with them: the odd function is called the hyperbolic sine of \(x\) and the even function is called the hyperbolic cosine of \(x\). They are denoted by

\[
\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}
\]

where \(\sinh\) is pronounced “cinch” and \(\cosh\) rhymes with “gosh.” From these two building blocks we can create four more functions to produce the following set of six hyperbolic functions.

<table>
<thead>
<tr>
<th>Hyperbolic sine</th>
<th>(\sinh x = \frac{e^x - e^{-x}}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbolic cosine</td>
<td>(\cosh x = \frac{e^x + e^{-x}}{2})</td>
</tr>
<tr>
<td>Hyperbolic tangent</td>
<td>(\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}})</td>
</tr>
<tr>
<td>Hyperbolic cotangent</td>
<td>(\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}})</td>
</tr>
<tr>
<td>Hyperbolic secant</td>
<td>(\sech x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}})</td>
</tr>
<tr>
<td>Hyperbolic cosecant</td>
<td>(\csch x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}})</td>
</tr>
</tbody>
</table>

#### Example 1

\[
\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1 - 1}{2} = 0
\]

\[
\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1 + 1}{2} = 1
\]

\[
\sinh 2 = \frac{e^2 - e^{-2}}{2} \approx 3.6269
\]
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**GRAPHS OF THE HYPERBOLIC FUNCTIONS**

The graphs of the hyperbolic functions, which are shown in Figure 6.9.1, can be generated with a graphing utility, but it is worthwhile to observe that the general shape of the graph of \( y = \cosh x \) can be obtained by sketching the graphs of \( y = \frac{1}{2}e^x \) and \( y = \frac{1}{2}e^{-x} \) separately and adding the corresponding \( y \)-coordinates [see part (a) of the figure]. Similarly, the general shape of the graph of \( y = \sinh x \) can be obtained by sketching the graphs of \( y = \frac{1}{2}e^x \) and \( y = -\frac{1}{2}e^{-x} \) separately and adding corresponding \( y \)-coordinates [see part (b) of the figure].

- **Figure 6.9.1**

Observe that \( \sinh x \) has a domain of \((-\infty, +\infty)\) and a range of \((-\infty, +\infty)\), whereas \( \cosh x \) has a domain of \((-\infty, +\infty)\) and a range of \([1, +\infty)\). Observe also that \( y = \frac{1}{2}e^x \) and \( y = \frac{1}{2}e^{-x} \) are *curvilinear asymptotes* for \( y = \cosh x \) in the sense that the graph of \( y = \cosh x \) gets closer and closer to the graph of \( y = \frac{1}{2}e^x \) as \( x \to +\infty \) and gets closer and closer to the graph of \( y = \frac{1}{2}e^{-x} \) as \( x \to -\infty \). (See Section 4.3.) Similarly, \( y = \frac{1}{2}e^x \) is a curvilinear asymptote for \( y = \sinh x \) as \( x \to +\infty \) and \( y = -\frac{1}{2}e^{-x} \) is a curvilinear asymptote as \( x \to -\infty \). Other properties of the hyperbolic functions are explored in the exercises.

**HANGING CABLES AND OTHER APPLICATIONS**

Hyperbolic functions arise in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also occur when a homogeneous, flexible cable is suspended between two points, as with a telephone line hanging between two poles. Such a cable forms a curve, called a *catenary* (from the Latin *catena*, meaning “chain”). If, as in Figure 6.9.2, a coordinate system is introduced so that the low point of the cable lies on the \( y \)-axis, then it can be shown using principles of physics that the cable has an equation of the form

\[
y = a \cosh \left( \frac{x}{a} \right) + c
\]
where the parameters $a$ and $c$ are determined by the distance between the poles and the composition of the cable.

## HYPERBOLIC IDENTITIES

The hyperbolic functions satisfy various identities that are similar to identities for trigonometric functions. The most fundamental of these is

$$\cosh^2 x - \sinh^2 x = 1 \quad (1)$$

which can be proved by writing

$$\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x)$$

$$= \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^x - e^{-x}}{2} \right)$$

$$= e^x \cdot e^{-x} = 1$$

Other hyperbolic identities can be derived in a similar manner or, alternatively, by performing algebraic operations on known identities. For example, if we divide (1) by $\cosh^2 x$, we obtain

$$1 - \tanh^2 x = \text{sech}^2 x$$

and if we divide (1) by $\sinh^2 x$, we obtain

$$\coth^2 x - 1 = \text{csch}^2 x$$

The following theorem summarizes some of the more useful hyperbolic identities. The proofs of those not already obtained are left as exercises.

### 6.9.2 THEOREM

| $\cosh x + \sinh x = e^x$ | $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$ |
| $\cosh x - \sinh x = e^{-x}$ | $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$ |
| $\cosh^2 x - \sinh^2 x = 1$ | $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$ |
| $1 - \tanh^2 x = \text{sech}^2 x$ | $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$ |
| $\coth^2 x - 1 = \text{csch}^2 x$ | $\sinh 2x = 2 \sinh x \cosh x$ |
| $\cosh(-x) = \cosh x$ | $\cosh 2x = \cosh^2 x + \sinh^2 x$ |
| $\sinh(-x) = -\sinh x$ | $\cosh 2x = 2 \sinh^2 x + 1 = 2 \cosh^2 x - 1$ |

### WHY THEY ARE CALLED HYPERBOLIC FUNCTIONS

Recall that the parametric equations

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)$$

represent the unit circle $x^2 + y^2 = 1$ (Figure 6.9.3a), as may be seen by writing

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

If $0 \leq t \leq 2\pi$, then the parameter $t$ can be interpreted as the angle in radians from the positive $x$-axis to the point $(\cos t, \sin t)$ or, alternatively, as twice the shaded area of the sector in Figure 6.9.3a (verify). Analogously, the parametric equations

$$x = \cosh t, \quad y = \sinh t \quad (-\infty < t < +\infty)$$

represent the catenary $y = a \cosh(x/a) + c$ (Figure 6.9.2), as may be seen by writing

$$y^2 = a^2 \left( \cosh^2 \frac{x}{a} - 1 \right)$$

Figure 6.9.2

A flexible cable suspended between two poles forms a catenary.
6.9 Hyperbolic Functions and Hanging Cables

represent a portion of the curve \(x^2 - y^2 = 1\), as may be seen by writing

\[x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1\]

and observing that \(x = \cosh t > 0\). This curve, which is shown in Figure 6.9.3b, is the right half of a larger curve called the *unit hyperbola*; this is the reason why the functions in this section are called *hyperbolic* functions. It can be shown that if \(t \geq 0\), then the parameter \(t\) can be interpreted as twice the shaded area in Figure 6.9.3b. (We omit the details.)

### DERIVATIVE AND INTEGRAL FORMULAS

Derivative formulas for \(\sinh x\) and \(\cosh x\) can be obtained by expressing these functions in terms of \(e^x\) and \(e^{-x}\):

\[
\frac{d}{dx} [\sinh x] = \cosh x \frac{d}{dx} \left[ \frac{e^x - e^{-x}}{2} \right] = \frac{e^x + e^{-x}}{2} = \cosh x
\]

\[
\frac{d}{dx} [\cosh x] = \sinh x \frac{d}{dx} \left[ \frac{e^x + e^{-x}}{2} \right] = \frac{e^x - e^{-x}}{2} = \sinh x
\]

Derivatives of the remaining hyperbolic functions can be obtained by expressing them in terms of \(\sinh\) and \(\cosh\) and applying appropriate identities. For example,

\[
\frac{d}{dx} [\tanh x] = \sech^2 x \frac{d}{dx} \left[ \frac{\sinh x}{\cosh x} \right] = \frac{\cosh x \frac{d}{dx} [\sinh x] - \sinh x \frac{d}{dx} [\cosh x]}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \sech^2 x
\]

The following theorem provides a complete list of the generalized derivative formulas and corresponding integration formulas for the hyperbolic functions.

#### 6.9.3 THEOREM

- \(\frac{d}{dx} [\sinh u] = \cosh u \frac{du}{dx}\)
- \(\frac{d}{dx} [\cosh u] = \sinh u \frac{du}{dx}\)
- \(\frac{d}{dx} [\tanh u] = \sec^2 u \frac{du}{dx}\)
- \(\frac{d}{dx} [\coth u] = -\csc^2 u \frac{du}{dx}\)
- \(\frac{d}{dx} [\sech u] = -\tanh u \frac{du}{dx}\)
- \(\frac{d}{dx} [\csch u] = -\coth u \frac{du}{dx}\)

\[\int \cosh u \, du = \sinh u + C\]

\[\int \sinh u \, du = \cosh u + C\]

\[\int \sec^2 u \, du = \tan u + C\]

\[\int -\csc^2 u \, du = -\cot u + C\]

\[\int -\tanh u \, du = -\sech u + C\]

\[\int -\coth u \, du = -\csch u + C\]

#### Example 2

\[
\frac{d}{dx} [\cosh(x^3)] = \sinh(x^3) \cdot \frac{d}{dx} [x^3] = 3x^2 \sinh(x^3)
\]

\[
\frac{d}{dx} [\ln(\tanh x)] = \frac{1}{\tanh x} \cdot \frac{d}{dx} [\tanh x] = \frac{\sech^2 x}{\tanh x}
\]
\section*{Example 3}

\begin{align*}
\int \sinh^5 x \cosh x \, dx &= \frac{1}{6} \sinh^6 x + C \\
\int \tanh x \, dx &= \int \frac{\sinh x}{\cosh x} \, dx \\
&= \ln |\cosh x| + C
\end{align*}

We were justified in dropping the absolute value signs since \( \cosh x > 0 \) for all \( x \).

\section*{Example 4}

A 100 ft wire is attached at its ends to the tops of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?

\textbf{Solution.} From above, the wire forms a catenary curve with equation

\[ y = a \cosh \left( \frac{x}{a} \right) + c \]

where the origin is on the ground midway between the poles. Using Formula (4) of Section 6.4 for the length of the catenary, we have

\begin{align*}
100 &= \int_{-45}^{45} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \\
&= 2 \int_{0}^{45} \sqrt{1 + \sinh^2 \left( \frac{x}{a} \right)} \, dx \\
&= 2 \int_{0}^{45} \cosh \left( \frac{x}{a} \right) \, dx \\
&= 2a \sinh \left( \frac{45}{a} \right)
\end{align*}

Using a calculating utility’s numeric solver to solve

\[ 100 = 2a \sinh \left( \frac{45}{a} \right) \]

for \( a \) gives \( a \approx 56.01 \). Then

\[ 50 = y(45) = 56.01 \cosh \left( \frac{45}{56.01} \right) + c \approx 75.08 + c \]

so \( c \approx -25.08 \). Thus, the middle of the wire is \( y(0) \approx 56.01 - 25.08 = 30.93 \) ft above the ground (Figure 6.9.4).

\section*{INVERSES OF HYPERBOLIC FUNCTIONS}

Referring to Figure 6.9.1, it is evident that the graphs of \( \sinh x, \tanh x, \coth x, \) and \( \text{csch} \, x \) pass the horizontal line test, but the graphs of \( \cosh x \) and \( \text{sech} \, x \) do not. In the latter case, restricting \( x \) to be nonnegative makes the functions invertible (Figure 6.9.5). The graphs of the six inverse hyperbolic functions in Figure 6.9.6 were obtained by reflecting the graphs of the hyperbolic functions (with the appropriate restrictions) about the line \( y = x \).
6.9 Hyperbolic Functions and Hanging Cables

Table 6.9.1 summarizes the basic properties of the inverse hyperbolic functions. You should confirm that the domains and ranges listed in this table agree with the graphs in Figure 6.9.6.

<table>
<thead>
<tr>
<th>FUNCTION</th>
<th>DOMAIN</th>
<th>RANGE</th>
<th>BASIC RELATIONSHIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>sinh⁻¹x</td>
<td>(-∞, +∞)</td>
<td>(-∞, +∞)</td>
<td>sinh⁻¹(sinh x) = x if -∞ &lt; x &lt; +∞</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>sinh⁻¹(x) = x if -∞ &lt; x &lt; +∞</td>
</tr>
<tr>
<td>cosh⁻¹x</td>
<td>[1, +∞)</td>
<td>[0, +∞)</td>
<td>cosh⁻¹(cosh x) = x if x ≥ 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>cosh⁻¹(x) = x if x ≥ 1</td>
</tr>
<tr>
<td>tanh⁻¹x</td>
<td>(-1, 1)</td>
<td>(-∞, +∞)</td>
<td>tanh⁻¹(tanh x) = x if -∞ &lt; x &lt; +∞</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>tanh⁻¹(x) = x if -1 &lt; x &lt; 1</td>
</tr>
<tr>
<td>coth⁻¹x</td>
<td>(-∞, -1) ∪ (1, +∞)</td>
<td>(-∞, 0) ∪ (0, +∞)</td>
<td>coth⁻¹(coth x) = x if x &lt; 0 or x &gt; 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>coth⁻¹(x) = x if x &lt; -1 or x &gt; 1</td>
</tr>
<tr>
<td>sech⁻¹x</td>
<td>(0, 1]</td>
<td>[0, +∞)</td>
<td>sech⁻¹(sech x) = x if x ≥ 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>sech⁻¹(x) = x if 0 &lt; x ≤ 1</td>
</tr>
<tr>
<td>csch⁻¹x</td>
<td>(-∞, 0) ∪ (0, +∞)</td>
<td>(-∞, 0) ∪ (0, +∞)</td>
<td>csch⁻¹(csch x) = x if x &lt; 0 or x &gt; 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>csch⁻¹(x) = x if x &lt; 0 or x &gt; 0</td>
</tr>
</tbody>
</table>

With the restriction that x ≥ 0, the curves y = cosh x and y = sech x pass the horizontal line test.

Figure 6.9.5

Figure 6.9.6
LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS

Because the hyperbolic functions are expressible in terms of \( e^t \), it should not be surprising that the inverse hyperbolic functions are expressible in terms of natural logarithms; the next theorem shows that this is so.

6.9.4 Theorem  

The following relationships hold for all \( x \) in the domains of the stated inverse hyperbolic functions:

\[
\begin{align*}
\sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}) \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) \\
\tanh^{-1} x &= \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) \\
\coth^{-1} x &= \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right) \\
\sech^{-1} x &= \ln \left( \frac{1 + \sqrt{1 - x^2}}{x} \right) \\
\csch^{-1} x &= \ln \left( \frac{1 + \sqrt{1 + x^2}}{|x|} \right)
\end{align*}
\]

We will show how to derive the first formula in this theorem and leave the rest as exercises. The basic idea is to write the equation \( x = \sinh y \) in terms of exponential functions and solve this equation for \( y \) as a function of \( x \). This will produce the equation \( y = \sinh^{-1} x \) with \( \sinh^{-1} x \) expressed in terms of natural logarithms. Expressing \( x = \sinh y \) in terms of exponentials yields

\[ x = \sinh y = \frac{e^y - e^{-y}}{2} \]

which can be rewritten as

\[ e^y - 2x - e^{-y} = 0 \]

Multiplying this equation through by \( e^y \) we obtain

\[ e^{2y} - 2xe^y - 1 = 0 \]

and applying the quadratic formula yields

\[ e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1} \]

Since \( e^y > 0 \), the solution involving the minus sign is extraneous and must be discarded. Thus,

\[ e^y = x + \sqrt{x^2 + 1} \]

Taking natural logarithms yields

\[ y = \ln(x + \sqrt{x^2 + 1}) \quad \text{or} \quad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \]

Example 5

\[ \sinh^{-1} 1 = \ln(1 + \sqrt{1^2 + 1}) = \ln(1 + \sqrt{2}) \approx 0.8814 \]

\[ \tanh^{-1} \left( \frac{1}{2} \right) = \frac{1}{2} \ln \left( \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right) = \frac{1}{2} \ln 3 \approx 0.5493 \]
### 6.9 Hyperbolic Functions and Hanging Cables

#### DERIVATIVES AND INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

Formulas for the derivatives of the inverse hyperbolic functions can be obtained from Theorem 6.9.4. For example,

\[
\frac{d}{dx} \left( \sinh^{-1} \frac{x}{x} \right) = \frac{1}{\sqrt{x^2 + 1}} \frac{d}{dx} \left( \ln(x + \sqrt{x^2 + 1}) \right) = \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}
\]

This computation leads to two integral formulas, a formula that involves \( \sinh^{-1} x \) and an equivalent formula that involves logarithms:

\[
\int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x + C = \ln(x + \sqrt{x^2 + 1}) + C
\]

The following two theorems list the generalized derivative formulas and corresponding integration formulas for the inverse hyperbolic functions. Some of the proofs appear as exercises.

#### 6.9.5 THEOREM

\[
\begin{align*}
\frac{d}{dx} (\sinh^{-1} u) &= \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}, \quad |u| < 1 \\
\frac{d}{dx} (\cosh^{-1} u) &= \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \quad u > 1 \\
\frac{d}{dx} (\tanh^{-1} u) &= \frac{1}{1 - u^2} \frac{du}{dx}, \quad |u| < 1 \\
\frac{d}{dx} (\tanh^{-1} u) &= \frac{1}{1 - u^2} \frac{du}{dx}, \quad |u| > 1
\end{align*}
\]

#### 6.9.6 THEOREM

If \( a > 0 \), then

\[
\begin{align*}
\int \frac{du}{\sqrt{a^2 + u^2}} &= \sinh^{-1} \left( \frac{u}{a} \right) + C \quad \text{or} \quad \ln(u + \sqrt{u^2 + a^2}) + C \\
\int \frac{du}{\sqrt{u^2 - a^2}} &= \cosh^{-1} \left( \frac{u}{a} \right) + C \quad \text{or} \quad \ln(u + \sqrt{u^2 - a^2}) + C, \quad u > a \\
\int \frac{du}{a^2 - u^2} &= \frac{1}{a} \tanh^{-1} \left( \frac{u}{a} \right) + C, \quad |u| < a \\
\int \frac{du}{a^2 - u^2} &= \frac{1}{a} \coth^{-1} \left( \frac{u}{a} \right) + C, \quad |u| > a \\
\int \frac{du}{a\sqrt{u^2 - a^2}} &= -\frac{1}{a} \sec^{-1} \left( \frac{|u|}{a} \right) + C \quad \text{or} \quad -\frac{1}{a} \ln \left( \frac{a + \sqrt{a^2 - u^2}}{|u|} \right) + C, \quad 0 < |u| < a \\
\int \frac{du}{u\sqrt{a^2 + u^2}} &= -\frac{1}{a} \csc^{-1} \left( \frac{|u|}{a} \right) + C \quad \text{or} \quad -\frac{1}{a} \ln \left( \frac{a + \sqrt{a^2 + u^2}}{|u|} \right) + C, \quad u \neq 0
\end{align*}
\]
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Example 6 Evaluate \( \int \frac{dx}{\sqrt{4x^2 - 9}}, \quad x > \frac{3}{2} \)

Solution. Let \( u = 2x \). Thus, \( du = 2 \, dx \) and

\[
\int \frac{dx}{\sqrt{4x^2 - 9}} = \frac{1}{2} \int \frac{2 \, dx}{\sqrt{4x^2 - 9}} = \frac{1}{2} \int \frac{du}{\sqrt{u^2 - 3^2}} = \frac{1}{2} \cosh^{-1} \left( \frac{u}{3} \right) + C = \frac{1}{2} \cosh^{-1} \left( \frac{2x}{3} \right) + C
\]

Alternatively, we can use the logarithmic equivalent of \( \cosh^{-1}(2x/3) \), \( \cosh^{-1} \left( \frac{2x}{3} \right) = \ln(2x + \sqrt{4x^2 - 9}) - \ln 3 \) (verify), and express the answer as

\[
\int \frac{dx}{\sqrt{4x^2 - 9}} = \frac{1}{2} \ln(2x + \sqrt{4x^2 - 9}) + C
\]

Quick Check Exercises 6.9 (See page 485 for answers.)

1. \( \cosh x = \ldots \), \( \sinh x = \ldots \)
   \( \tanh x = \ldots \)

2. Complete the table.

<table>
<thead>
<tr>
<th>( \cosh x )</th>
<th>( \sinh x )</th>
<th>( \tanh x )</th>
<th>( \coth x )</th>
<th>( \sech x )</th>
<th>( \csch x )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>DOMAIN</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>RANGE</strong></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

3. The parametric equations

\( x = \cosh t, \quad y = \sinh t \quad (\infty < t < +\infty) \)

represent the right half of the curve called a _______. Eliminating the parameter, the equation of this curve is _______.

Exercise Set 6.9 Graphing Utility

1–2 Approximate the expression to four decimal places.

1. (a) \( \sinh 3 \) \quad (b) \( \cosh(-2) \) \quad (c) \( \tanh(\ln 4) \)
   (d) \( \csch(-2) \) \quad (e) \( \cosh^{-1} 3 \) \quad (f) \( \tanh^{-1} \frac{3}{4} \)

2. (a) \( \csch(-1) \) \quad (b) \( \sech(\ln 2) \) \quad (c) \( \coth 1 \)
   (d) \( \cosh^{-1} \frac{1}{2} \) \quad (e) \( \coth^{-1} 3 \) \quad (f) \( \csch^{-1}(-\sqrt{3}) \)

3. Find the exact numerical value of each expression.
   (a) \( \sinh(\ln 3) \) \quad (b) \( \cosh(-\ln 2) \)
   (c) \( \tanh(2 \ln 5) \) \quad (d) \( \sinh(-3 \ln 2) \)

4. In each part, rewrite the expression as a ratio of polynomials.
   (a) \( \cosh(\ln x) \) \quad (b) \( \sinh(\ln x) \)
   (c) \( \tanh(2 \ln x) \) \quad (d) \( \cosh(-\ln x) \)

5. In each part, a value for one of the hyperbolic functions is given at an unspecified positive number \( x_0 \). Use appropriate identities to find the exact values of the remaining five hyperbolic functions at \( x_0 \).
   (a) \( \sinh x_0 = 2 \) \quad (b) \( \cosh x_0 = \frac{3}{4} \) \quad (c) \( \tanh x_0 = \frac{5}{3} \)

6. Obtain the derivative formulas for \( \csch x, \sech x, \) and \( \coth x \) from the derivative formulas for \( \sinh x, \cosh x, \) and \( \tanh x \).

7. Find the derivatives of \( \cosh^{-1} x \) and \( \tanh^{-1} x \) by differentiating the formulas in Theorem 6.9.4.

8. Find the derivatives of \( \sinh^{-1} x, \cosh^{-1} x, \) and \( \tanh^{-1} x \) by differentiating the equations \( x = \sinh y, x = \cosh y, \) and \( x = \tanh y \) implicitly.

9–28 Find \( dy/dx \).

9. \( y = \sinh(4x - 8) \) \quad 10. \( y = \cosh(4x) \)
11. \( y = \coth(\ln x) \)
12. \( y = \ln(\tanh 2x) \)
13. \( y = \csch(1/x) \)
14. \( y = \sech(e^{2x}) \)
15. \( y = \sqrt{4x + \cosh^2(3x)} \)
16. \( y = \sinh^3(2x) \)
17. \( y = x^3 \tanh^2(\sqrt{x}) \)
18. \( y = \sinh(3x) \)
19. \( y = \sinh^{-1}(\frac{1}{2}x) \)
20. \( y = \sinh^{-1}(\frac{1}{x}) \)
21. \( y = \ln(\cosh^{-1}x) \)
22. \( y = \cosh^{-1}(\sinh^{-1}x) \)
23. \( y = \frac{1}{\tanh^{-1}x} \)
24. \( y = (\cosh^{-1}x)^2 \)
25. \( y = \cosh^{-1}(\cosh(x)) \)
26. \( y = \sinh^{-1}(\tanh x) \)
27. \( y = e^x \sech^{-1}\sqrt{x} \)
28. \( y = (1 + x \csch^{-1}x)^{10} \)

29–44 Evaluate the integrals.
29. \( \int \sinh^5 x \cosh x \, dx \)
30. \( \int \cosh(2x - 3) \, dx \)
31. \( \int \sqrt{\tanh x} \sech^2 x \, dx \)
32. \( \int \cosh^3(3x) \, dx \)
33. \( \int \tanh x \, dx \)
34. \( \int \coth^2 x \sech^2 x \, dx \)
35. \( \int_{\ln 3}^{\ln 2} \tanh x \sech^3 x \, dx \)
36. \( \int_{0}^{\ln 3} e^x - e^{-x} \, dx \)
37. \( \int \frac{dx}{\sqrt{1 + 9x^2}} \)
38. \( \int \frac{dx}{\sqrt{x^2 - 2}} \quad (x > \sqrt{2}) \)
39. \( \int \frac{dx}{\sqrt{1 - e^{2x}}} \quad (x < 0) \)
40. \( \int \frac{dx}{\sqrt{1 + \cos \theta}} \)
41. \( \int \frac{dx}{x \sqrt{1 + 4x^2}} \)
42. \( \int \frac{dx}{\sqrt{9x^2 - 25}} \quad (x > 5/3) \)
43. \( \int_{1/2}^{\sqrt{3}} \frac{dx}{1 - x^2} \)
44. \( \int_{0}^{\sqrt{3}} \frac{dt}{\sqrt{t^2 + 1}} \)

45–48 True–False Determine whether the statement is true or false. Explain your answer.
45. The equation \( \cosh x = \sinh x \) has no solutions.
46. Exactly two of the hyperbolic functions are bounded.
47. There is exactly one hyperbolic function \( f(x) \) such that for all real numbers \( a \), the equation \( f(x) = a \) has a unique solution \( x \).
48. The identities in Theorem 6.9.2 may be obtained from the corresponding trigonometric identities by replacing each trigonometric function with its hyperbolic analogue.
49. Find the area enclosed by \( y = \sinh 2x \), \( y = 0 \), and \( x = \ln 3 \).
50. Find the volume of the solid that is generated when the region enclosed by \( y = \sech x \), \( y = 0 \), \( x = 0 \), and \( x = \ln 2 \) is revolved about the \( x \)-axis.
51. Find the volume of the solid that is generated when the region enclosed by \( y = \cosh 2x \), \( y = \sinh 2x \), \( x = 0 \), and \( x = 5 \) is revolved about the \( x \)-axis.
52. Approximate the positive value of the constant \( a \) such that the area enclosed by \( y = \cosh ax \), \( y = 0 \), \( x = 0 \), and \( x = 1 \) is 2 square units. Express your answer to at least five decimal places.
53. Find the arc length of the catenary \( y = \cosh x \) between \( x = 0 \) and \( x = \ln 2 \).
54. Find the arc length of the catenary \( y = a \cosh(x/a) \) between \( x = 0 \) and \( x = x_1 \) (\( x_1 > 0 \)).
55. In parts (a)–(f) find the limits, and confirm that they are consistent with the graphs in Figures 6.9.1 and 6.9.6.
   (a) \( \lim_{x \to +\infty} \sinh x \)
   (b) \( \lim_{x \to -\infty} \cosh x \)
   (c) \( \lim_{x \to +\infty} \tanh x \)
   (d) \( \lim_{x \to -\infty} \tanh x \)
   (e) \( \lim_{x \to +\infty} \sinh^{-1} x \)
   (f) \( \lim_{x \to +\infty} \tanh^{-1} x \)

56. Explain how to obtain the asymptotes for \( y = \tanh x \) from the curvilinear asymptotes for \( y = \cosh x \) and \( y = \sinh x \).
57. Prove that \( \sinh x \) is an odd function of \( x \) and that \( \cosh x \) is an even function of \( x \), and check that this is consistent with the graphs in Figure 6.9.1.

58–59 Prove the identities.
58. (a) \( \cosh x + \sinh x = e^x \)
   (b) \( \cosh x - \sinh x = e^{-x} \)
   (c) \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \)
   (d) \( \sinh 2x = 2 \sinh x \cosh x \)
   (e) \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \)
   (f) \( \cosh 2x = \cosh^2 x + \sinh^2 x \)
   (g) \( \cosh 2x = 2 \sinh^2 x + 1 \)
   (h) \( \cosh 2x = 2 \cosh^2 x - 1 \)
59. (a) \( 1 - \tanh^2 x = \sech^2 x \)
   (b) \( \tanh(x + y) = \frac{\tan x + \tan y}{1 + \tan x \tan y} \)
   (c) \( \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \)

60. Prove:
   (a) \( \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \), \( x \geq 1 \)
   (b) \( \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) \), \( -1 < x < 1 \).
61. Use Exercise 60 to obtain the derivative formulas for \( \cosh^{-1} x \) and \( \tanh^{-1} x \).
62. Prove:
   \[\sech^{-1} x = \cosh^{-1}(1/x) , \quad 0 < x \leq 1\]
   \[\coth^{-1} x = \tanh^{-1}(1/x) , \quad |x| > 1\]
   \[\csch^{-1} x = \sinh^{-1}(1/x) , \quad x \neq 0\]
63. Use Exercise 62 to express the integral
   \[\int \frac{du}{1 - u^2}\]
   entirely in terms of \( \tanh^{-1} x \).
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64. Show that
   (a) \( \frac{d}{dx} \text{sech}^{-1}|x| = -\frac{1}{x \sqrt{1 - x^2}} \)
   (b) \( \frac{d}{dx} \text{csch}^{-1}|x| = -\frac{1}{x \sqrt{1 + x^2}} \)

65. In each part, find the limit.
   (a) \( \lim_{x \to a} (\text{csch}^{-1} x - \ln x) \)
   (b) \( \lim_{x \to a} \frac{\cosh x}{e^x} \)

66. Use the first and second derivatives to show that the graph of \( y = \tanh^{-1} x \) is always increasing and has an inflection point at the origin.

67. The integration formulas for \( 1/\sqrt{u^2 - a^2} \) in Theorem 6.9.6 are valid for \( a > 0 \). Show that the following formula is valid for \( u < -a \):
   \[ \int \frac{du}{\sqrt{u^2 - a^2}} = -\cosh^{-1} \left( \frac{u}{a} \right) + C \text{ or } \ln \left| a + \sqrt{u^2 - a^2} \right| + C \]

68. Show that \( (\sinh x + \cosh x)^n = \sinh nx + \cosh nx \).

69. Show that
   \[ \int_{-a}^{a} e^x \, dx = \frac{2 \sinh at}{t} \]

70. A cable is suspended between two poles as shown in Figure 6.9.2. Assume that the equation of the curve formed by the cable is \( y = a \cosh(x/a) \), where \( a \) is a positive constant. Suppose that the \( x \)-coordinates of the points of support are \( x = -b \) and \( x = b \), where \( b > 0 \).
   (a) Show that the length \( L \) of the cable is given by
   \[ L = 2a \sinh \frac{b}{a} \]
   (b) Show that the sag \( S \) (the vertical distance between the highest and lowest points on the cable) is given by
   \[ S = a \cosh \frac{b}{a} - a \]

71–72 These exercises refer to the hanging cable described in Exercise 70.

71. Assuming that the poles are 400 ft apart and the sag in the cable is 30 ft, approximate the length of the cable by approximating \( a \). Express your final answer to the nearest tenth of a foot. [Hint: First let \( u = 200/a \).]

72. Assuming that the cable is 120 ft long and the poles are 100 ft apart, approximate the sag in the cable by approximating \( a \). Express your final answer to the nearest tenth of a foot. [Hint: First let \( u = 50/a \).]

73. The design of the Gateway Arch in St. Louis, Missouri, by architect Eero Saarinen was implemented using equations provided by Dr. Hannskarl Badel. The equation used for the centerline of the arch was
   \[ y = 693.8597 - 68.7672 \cosh(0.01000333x) \text{ ft} \]
   for \( x \) between \(-299.2239 \) and \( 299.2239 \).
   (a) Use a graphing utility to graph the centerline of the arch.
   (b) Find the length of the centerline to four decimal places.
   (c) For what values of \( x \) is the height of the arch 100 ft? Round your answers to four decimal places.
   (d) Approximate, to the nearest degree, the acute angle that the tangent line to the centerline makes with the ground at the ends of the arch.

74. Suppose that a hollow tube rotates with a constant angular velocity of \( \omega \text{ rad/s} \) about a horizontal axis at one end of the tube, as shown in the accompanying figure. Assume that an object is free to slide without friction in the tube while the tube is rotating. Let \( r \) be the distance from the object to the pivot point at time \( t \geq 0 \), and assume that the object is at rest and \( r = 0 \) when \( t = 0 \). It can be shown that if the tube is horizontal at time \( t = 0 \) and rotating as shown in the figure, then
   \[ r = \frac{g}{2\omega^2} \left[ \sinh(\omega t) - \sin(\omega t) \right] \]
   during the period that the object is in the tube. Assume that \( t \) in seconds and \( r \) in meters, and use \( g = 9.8 \text{ m/s}^2 \) and \( \omega = 2 \text{ rad/s} \).
   (a) Graph \( r \) versus \( t \) for \( 0 \leq t \leq 1 \).
   (b) Assuming that the tube has a length of 1 m, approximately how long does it take for the object to reach the end of the tube?
   (c) Use the result of part (b) to approximate \( dr/dt \) at the instant that the object reaches the end of the tube.

75. The accompanying figure (on the next page) shows a person pulling a boat by holding a rope of length \( a \) attached to the bow and walking along the edge of a dock. If we assume that the rope is always tangent to the curve traced by the bow of the boat, then this curve, which is called a tractrix, has the property that the segment of the tangent line to the centerline makes with the ground at the ends of the arch.
   \[ y = a \sech^{-1} \frac{x}{a} - \sqrt{a^2 - x^2} \]
   (a) Show that to move the bow of the boat to a point \( (x, y) \), the person must walk a distance
   \[ D = a \sech^{-1} \frac{x}{a} \]
   from the origin.
   (b) If the rope has a length of 15 m, how far must the person walk from the origin to bring the boat 10 m from the dock? Round your answer to two decimal places.
   (c) Find the distance traveled by the bow along the tractrix as it moves from its initial position to the point where it is 5 m from the dock.
76. Writing Suppose that, by analogy with the trigonometric functions, we define \( \cosh t \) and \( \sinh t \) geometrically using Figure 6.9.3b:

“For any real number \( t \), define \( x = \cosh t \) and \( y = \sinh t \) to be the unique values of \( x \) and \( y \) such that

\[
\cosh^2 t - \sinh^2 t = 1.
\]

QUICK CHECK ANSWERS 6.9

1. \( \frac{e^t + e^{-t}}{2} \); \( \frac{e^t - e^{-t}}{2} \); \( \frac{e^t - e^{-t}}{e^t + e^{-t}} \)

2. \( \begin{array}{cccccc}
\text{cosh } x & \text{sinh } x & \text{tanh } x & \text{coth } x & \text{sech } x & \text{csch } x \\
\text{DOMAIN} & (-\infty, +\infty) & (-\infty, +\infty) & (-\infty, 0) \cup (0, +\infty) & (-\infty, +\infty) & (-\infty, 0) \cup (0, +\infty) \\
\text{RANGE} & [1, +\infty) & (-\infty, +\infty) & (-1, 1) & (-\infty, -1) \cup (1, +\infty) & (0, 1) & (-\infty, 0) \cup (0, +\infty)
\end{array} \)

3. unit hyperbola; \( x^2 - y^2 = 1 \)

4. \( \sinh x; \cosh x; \sech^2 x \)

5. \( \sinh x + C; \cosh x + C; \ln(\cosh x) + C \)

6. \( \frac{1}{\sqrt{x^2 - 1}}; \frac{1}{\sqrt{1 + x^2}}; \frac{1}{1 - x^2} \)

CHAPTER 6 REVIEW EXERCISES

1. Describe the method of slicing for finding volumes, and use that method to derive an integral formula for finding volumes by the method of disks.

2. State an integral formula for finding a volume by the method of cylindrical shells, and use Riemann sums to derive the formula.

3. State an integral formula for finding the arc length of a smooth curve \( y = f(x) \) over an interval \([a, b] \), and use Riemann sums to derive the formula.

4. State an integral formula for the work \( W \) done by a variable force \( F(x) \) applied in the direction of motion to an object moving from \( x = a \) to \( x = b \), and use Riemann sums to derive the formula.

5. State an integral formula for the fluid force \( F \) exerted on a vertical flat surface immersed in a fluid of weight density \( \rho \), and use Riemann sums to derive the formula.

6. Let \( R \) be the region in the first quadrant enclosed by \( y = x^2 \), \( y = 2 + x \), and \( x = 0 \). In each part, set up, but do not evaluate, an integral or a sum of integrals that will solve the problem.

(a) Find the area of \( R \) by integrating with respect to \( x \).

(b) Find the area of \( R \) by integrating with respect to \( y \).

(c) Find the volume of the solid generated by revolving \( R \) about the \( x \)-axis, integrating with respect to \( x \).

(d) Find the volume of the solid generated by revolving \( R \) about the \( x \)-axis, integrating with respect to \( y \).

(e) Find the volume of the solid generated by revolving \( R \) about the \( y \)-axis, integrating with respect to \( x \).

(f) Find the volume of the solid generated by revolving \( R \) about the \( y \)-axis, integrating with respect to \( y \).

(g) Find the volume of the solid generated by revolving \( R \) about the line \( y = -3 \) by integrating with respect to \( x \).

(h) Find the volume of the solid generated by revolving \( R \) about the line \( x = 5 \) by integrating with respect to \( x \).

7. Set up a sum of definite integrals that represents the total shaded area between the curves \( y = f(x) \) and \( y = g(x) \) in the accompanying figure on the next page.
8. The accompanying figure shows velocity versus time curves for two cars that move along a straight track, accelerating from rest at a common starting line.
(a) How far apart are the cars after 60 seconds?
(b) How far apart are the cars after \( T \) seconds, where \( 0 \leq T \leq 60 \) ?

9. Let \( R \) be the region enclosed by the curves \( y = x^2 + 4 \), \( y = x^3 \), and the \( y \)-axis. Find and evaluate a definite integral that represents the volume of the solid generated by revolving \( R \) about the \( x \)-axis.

10. A football has the shape of the solid generated by revolving the region bounded by the \( x \)-axis and the parabola \( y = 4R(x^2 - \frac{1}{4}L^2)/L^2 \) about the \( x \)-axis. Find its volume.

11. Find the volume of the solid whose base is the region bounded between the curves \( y = \sqrt{x} \) and \( y = 1/\sqrt{x} \) for \( 1 \leq x \leq 4 \) and whose cross sections perpendicular to the \( x \)-axis are squares.

12. Consider the region enclosed by \( y = \sin^{-1} x \), \( y = 0 \), and \( x = 1 \). Set up, but do not evaluate, an integral that represents the volume of the solid generated by revolving the region about the \( x \)-axis using (a) disks (b) cylindrical shells.

13. Find the arc length in the second quadrant of the curve \( x^{2/3} + y^{2/3} = 4 \) from \( x = -8 \) to \( x = -1 \).

14. Let \( C \) be the curve \( y = e^x \) between \( x = 0 \) and \( x = \ln 10 \). In each part, set up, but do not evaluate, an integral that solves the problem.
(a) Find the arc length of \( C \) by integrating with respect to \( x \).
(b) Find the arc length of \( C \) by integrating with respect to \( y \).

15. Find the area of the surface generated by revolving the curve \( y = \sqrt{2x-x^2} \), \( 9 \leq x \leq 16 \), about the \( x \)-axis.

16. Let \( C \) be the curve \( 27x - y^3 = 0 \) between \( y = 0 \) and \( y = 2 \). In each part, set up, but do not evaluate, an integral or a sum of integrals that solves the problem.
(a) Find the area of the surface generated by revolving \( C \) about the \( x \)-axis by integrating with respect to \( x \).
(b) Find the area of the surface generated by revolving \( C \) about the \( y \)-axis by integrating with respect to \( y \).
(c) Find the area of the surface generated by revolving \( C \) about the line \( y = -2 \) by integrating with respect to \( y \).

17. Consider the solid generated by revolving the region enclosed by \( y = \sec x \), \( x = 0 \), \( x = \pi/3 \), and \( y = 0 \) about the \( x \)-axis. Find the average value of the area of a cross section of this solid taken perpendicular to the \( x \)-axis.

18. Consider the solid generated by revolving the region enclosed by \( y = \sqrt{a^2 - x^2} \) and \( y = 0 \) about the \( x \)-axis. Without performing an integration, find the average value of the area of a cross section of this solid taken perpendicular to the \( x \)-axis.

19. (a) A spring exerts a force of 0.5 N when stretched 0.25 m beyond its natural length. Assuming that Hooke’s law applies, how much work was performed in stretching the spring to this length?
(b) How far beyond its natural length can the spring be stretched with 25 J of work?

20. A boat is anchored so that the anchor is 150 ft below the surface of the water. In the water, the anchor weighs 2000 lb and the chain weighs 30 lb/ft. How much work is required to raise the anchor to the surface?

21–22 Find the centroid of the region. ■
21. The region bounded by \( y^2 = 4x \) and \( y^2 = 8(x - 2) \).
22. The upper half of the ellipse \( (x/a)^2 + (y/b)^2 = 1 \).

23. In each part, set up, but do not evaluate, an integral that solves the problem.
(a) Find the fluid force exerted on a side of a box that has a 3 m square base and is filled to a depth of 1 m with a liquid of weight density \( \rho \) \( N/m^3 \).
(b) Find the fluid force exerted by a liquid of weight density \( \rho \) \( lb/ft^3 \) on a face of the vertical plate shown in part (a) of the accompanying figure.
(c) Find the fluid force exerted on the parabolic dam in part (b) of the accompanying figure by water that extends to the top of the dam.

24. Show that for any constant \( a \), the function \( y = \sinh(a x) \) satisfies the equation \( y'' = a^2 y \).

25. In each part, prove the identity.
(a) \( \cosh 3x = 4 \cosh^3 x - 3 \cosh x \)
(b) \( \cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x + 1)} \)
(c) \( \sinh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x - 1)} \)
CHAPTER 6 MAKING CONNECTIONS

1. Suppose that $f$ is a nonnegative function defined on $[0, 1]$ such that the area between the graph of $f$ and the interval $[0, 1]$ is $A_1$ and such that the area of the region $R$ between the graph of $g(x) = f(x^2)$ and the interval $[0, 1]$ is $A_2$. In each part, express your answer in terms of $A_1$ and $A_2$.
   (a) What is the volume of the solid of revolution generated by revolving $R$ about the y-axis?
   (b) Find a value of $a$ such that if the $xy$-plane were horizontal, the region $R$ would balance on the line $x = a$.

2. A water tank has the shape of a conical frustum with radius of the base 5 ft, radius of the top 10 ft and (vertical) height 15 ft. Suppose the tank is filled with water and consider the problem of finding the work required to pump all the water out through a hole in the top of the tank.
   (a) Solve this problem using the method of Example 5 in Section 6.6.
   (b) Solve this problem using Definition 6.6.3. [Hint: Think of the base as the head of a piston that expands to a water-tight fit against the sides of the tank as the piston is pushed upward. What important result about water pressure do you need to use?]

3. A disk of radius $a$ is an inhomogeneous lamina whose density is a function $f(r)$ of the distance $r$ to the center of the lamina. Modify the argument used to derive the method of cylindrical shells to find a formula for the mass of the lamina.

4. Compare Formula (10) in Section 6.7 with Formula (8) in Section 6.8. Then give a plausible argument that the force on a flat surface immersed vertically in a fluid of constant weight density is equal to the product of the area of the surface and the pressure at the centroid of the surface. Conclude that the force on the surface is the same as if the surface were immersed horizontally at the depth of the centroid.

5. Archimedes’ Principle states that a solid immersed in a fluid experiences a buoyant force equal to the weight of the fluid displaced by the solid.
   (a) Use the results of Section 6.8 to verify Archimedes’ Principle in the case of (i) a box-shaped solid with a pair of faces parallel to the surface of the fluid, (ii) a solid cylinder with vertical axis, and (iii) a cylindrical shell with vertical axis.
   (b) Give a plausible argument for Archimedes’ Principle in the case of a solid of revolution immersed in fluid such that the axis of revolution of the solid is vertical. [Hint: Approximate the solid by a union of cylindrical shells and use the result from part (a).]
The floating roof on the Stade de France sports complex is an ellipse. Finding the arc length of an ellipse involves numerical integration techniques introduced in this chapter.

In earlier chapters we obtained many basic integration formulas as an immediate consequence of the corresponding differentiation formulas. For example, knowing that the derivative of \( \sin x \) is \( \cos x \) enabled us to deduce that the integral of \( \cos x \) is \( \sin x \). Subsequently, we expanded our integration repertoire by introducing the method of \( u \)-substitution. That method enabled us to integrate many functions by transforming the integrand of an unfamiliar integral into a familiar form. However, \( u \)-substitution alone is not adequate to handle the wide variety of integrals that arise in applications, so additional integration techniques are still needed. In this chapter we will discuss some of those techniques, and we will provide a more systematic procedure for attacking unfamiliar integrals. We will talk more about numerical approximations of definite integrals, and we will explore the idea of integrating over infinite intervals.

7.1 AN OVERVIEW OF INTEGRATION METHODS

In this section we will give a brief overview of methods for evaluating integrals, and we will review the integration formulas that were discussed in earlier sections.

METHODS FOR APPROACHING INTEGRATION PROBLEMS

There are three basic approaches for evaluating unfamiliar integrals:

- **Technology**—CAS programs such as Mathematica, Maple, and the open source program Sage are capable of evaluating extremely complicated integrals, and such programs are increasingly available for both computers and handheld calculators.

- **Tables**—Prior to the development of CAS programs, scientists relied heavily on tables to evaluate difficult integrals arising in applications. Such tables were compiled over many years, incorporating the skills and experience of many people. One such table appears in the endpapers of this text, but more comprehensive tables appear in various reference books such as the *CRC Standard Mathematical Tables and Formulae*, CRC Press, Inc., 2002.

- **Transformation Methods**—Transformation methods are methods for converting unfamiliar integrals into familiar integrals. These include \( u \)-substitution, algebraic manipulation of the integrand, and other methods that we will discuss in this chapter.
7.1 An Overview of Integration Methods

None of the three methods is perfect; for example, CAS programs often encounter integrals that they cannot evaluate and they sometimes produce answers that are unnecessarily complicated, tables are not exhaustive and may not include a particular integral of interest, and transformation methods rely on human ingenuity that may prove to be inadequate in difficult problems.

In this chapter we will focus on transformation methods and tables, so it will not be necessary to have a CAS such as Mathematica, Maple, or Sage. However, if you have a CAS, then you can use it to confirm the results in the examples, and there are exercises that are designed to be solved with a CAS. If you have a CAS, keep in mind that many of the algorithms that it uses are based on the methods we will discuss here, so an understanding of these methods will help you to use your technology in a more informed way.

A REVIEW OF FAMILIAR INTEGRATION FORMULAS

The following is a list of basic integrals that we have encountered thus far:

CONSTANTS, POWERS, EXPONENTIALS

1. \[\int du = u + C\]
2. \[\int a \, du = a \int du = au + C\]
3. \[\int u^r \, du = \frac{u^{r+1}}{r+1} + C, \quad r \neq -1\]
4. \[\int \frac{du}{u} = \ln |u| + C\]
5. \[\int e^u \, du = e^u + C\]
6. \[\int b^u \, du = \frac{b^u}{\ln b} + C, \quad b > 0, \quad b \neq 1\]

TRIGONOMETRIC FUNCTIONS

7. \[\int \sin u \, du = -\cos u + C\]
8. \[\int \cos u \, du = \sin u + C\]
9. \[\int \sec^2 u \, du = \tan u + C\]
10. \[\int \csc^2 u \, du = -\cot u + C\]
11. \[\int \sec u \tan u \, du = \sec u + C\]
12. \[\int \csc u \cot u \, du = -\csc u + C\]
13. \[\int \tan u \, du = -\ln |\cos u| + C\]
14. \[\int \cot u \, du = \ln |\sin u| + C\]

HYPERBOLIC FUNCTIONS

15. \[\int \sinh u \, du = \cosh u + C\]
16. \[\int \cosh u \, du = \sinh u + C\]
17. \[\int \sech^2 u \, du = \tanh u + C\]
18. \[\int \csch^2 u \, du = -\coth u + C\]
19. \[\int \sech u \tanh u \, du = -\sech u + C\]
20. \[\int \csch u \coth u \, du = -\csch u + C\]

ALGEBRAIC FUNCTIONS (a > 0)

21. \[\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C \quad (|u| < a)\]
22. \[\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C\]
23. \[\int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{1}{a} \sec^{-1} \frac{|u|}{a} + C \quad (0 < a < |u|)\]
24. \[ \int \frac{du}{\sqrt{a^2 + u^2}} = \ln(u + \sqrt{u^2 + a^2}) + C \]

25. \[ \int \frac{du}{\sqrt{u^2 - a^2}} = \ln \left| u + \sqrt{u^2 - a^2} \right| + C \quad (0 < |u| < a) \]

26. \[ \int \frac{du}{u^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a + u}{a - u} \right| + C \]

27. \[ \int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C \quad (0 < |u| < a) \]

28. \[ \int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 + u^2}}{u} \right| + C \]

**REMARK**  Formula 25 is a generalization of a result in Theorem 6.9.6. Readers who did not cover Section 6.9 can ignore Formulas 24–28 for now, since we will develop other methods for obtaining them in this chapter.

**QUICK CHECK EXERCISES 7.1** (See page 491 for answers.)

1. Use algebraic manipulation and (if necessary) \( u \)-substitution to integrate the function.
   (a) \[ \int \frac{x + 1}{x} \, dx = \] 
   (b) \[ \int \frac{x + 2}{x + 1} \, dx = \] 
   (c) \[ \int \frac{2x + 1}{x^2 + 1} \, dx = \] 
   (d) \[ \int xe^{3\ln x} \, dx = \]

2. Use trigonometric identities and (if necessary) \( u \)-substitution to integrate the function.
   (a) \[ \int \frac{1}{\sec x} \, dx = \] 
   (b) \[ \int \frac{1}{\cos^2 x} \, dx = \] 
   (c) \[ \int (\cos^2 x + 1) \, dx = \] 
   (d) \[ \int \frac{1}{\sec x + \tan x} \, dx = \]

3. Integrate the function.
   (a) \[ \int \sqrt{x + 1} \, dx = \]
   (b) \[ \int e^{2x+1} \, dx = \]
   (c) \[ \int (\sin^3 x \cos x + \sin x \cos^3 x) \, dx = \]
   (d) \[ \int \frac{1}{(e^x + e^{-x})^2} \, dx = \]

**EXERCISE SET 7.1**

1–30 Evaluate the integrals by making appropriate \( u \)-substitutions and applying the formulas reviewed in this section.

1. \[ \int (4 - 2x)^3 \, dx \] 
2. \[ \int 3\sqrt{4 + 2x} \, dx \] 
3. \[ \int x \sec^2 (x^2) \, dx \] 
4. \[ \int 4x \tan(x^2) \, dx \] 
5. \[ \int \frac{\sin 3x}{2 + \cos 3x} \, dx \] 
6. \[ \int \frac{1}{9 + 4x^2} \, dx \] 
7. \[ \int e^x \sinh(e^x) \, dx \] 
8. \[ \int \frac{\sec(ln \, x) \tan(ln \, x)}{x} \, dx \] 
9. \[ \int e^{\tan x} \sec^2 x \, dx \] 
10. \[ \int \frac{x}{\sqrt{1 - x^2}} \, dx \] 
11. \[ \int \cos^2 x \sin 5x \, dx \] 
12. \[ \int \frac{\cos x}{\sin x \sqrt{\sin^2 x + 1}} \, dx \] 
13. \[ \int \frac{e^x}{\sqrt{4 + e^{2x}}} \, dx \] 
14. \[ \int \frac{e^{\tan^{-1} x}}{1 + x^2} \, dx \] 
15. \[ \int \frac{e^{\ln x}}{\sqrt{x - 1}} \, dx \] 
16. \[ \int (x + 1) \cot(x^2 + 2x) \, dx \] 
17. \[ \int \frac{\cosh \sqrt{x}}{\sqrt{x}} \, dx \] 
18. \[ \int \frac{dx}{x (\ln x)^2} \]
7.2 Integration by Parts

In this section we will discuss an integration technique that is essentially an antiderivative formulation of the formula for differentiating a product of two functions.

THE PRODUCT RULE AND INTEGRATION BY PARTS

Our primary goal in this section is to develop a general method for attacking integrals of the form

$$\int f(x)g(x) \, dx$$

As a first step, let $G(x)$ be any antiderivative of $g(x)$. In this case $G'(x) = g(x)$, so the product rule for differentiating $f(x)G(x)$ can be expressed as

$$\frac{d}{dx} [f(x)G(x)] = f(x)G'(x) + f'(x)G(x) = f(x)g(x) + f'(x)G(x) \quad (1)$$

This implies that $f(x)G(x)$ is an antiderivative of the function on the right side of (1), so we can express (1) in integral form as

$$\int [f(x)g(x) + f'(x)G(x)] \, dx = f(x)G(x)$$
or, equivalently, as

\[ \int f(x)g(x) \, dx = f(x)G(x) - \int f'(x)G(x) \, dx \] (2)

This formula allows us to integrate \( f(x)g(x) \) by integrating \( f'(x)G(x) \) instead, and in many cases the net effect is to replace a difficult integration with an easier one. The application of this formula is called integration by parts.

In practice, we usually rewrite (2) by letting

\( u = f(x), \quad du = f'(x) \, dx \)
\( v = G(x), \quad dv = G'(x) \, dx = g(x) \, dx \)

This yields the following alternative form for (2):

\[ \int u \, dv = uv - \int v \, du \] (3)

**Example 1** Use integration by parts to evaluate \( \int x \cos x \, dx \).

Solution. We will apply Formula (3). The first step is to make a choice for \( u \) and \( dv \) to put the given integral in the form \( \int u \, dv \). We will let

\( u = x \quad \text{and} \quad dv = \cos x \, dx \)

(Other possibilities will be considered later.) The second step is to compute \( du \) from \( u \) and \( v \) from \( dv \). This yields

\( du = dx \quad \text{and} \quad v = \int \cos x \, dx = \sin x \)

The third step is to apply Formula (3). This yields

\[ \int x \cos x \, dx = x \sin x - \int \sin x \, dx \]

\[ = x \sin x - (- \cos x) + C = x \sin x + \cos x + C \]

**GUIDELINES FOR INTEGRATION BY PARTS**

The main goal in integration by parts is to choose \( u \) and \( dv \) to obtain a new integral that is easier to evaluate than the original. In general, there are no hard and fast rules for doing this; it is mainly a matter of experience that comes from lots of practice. A strategy that often works is to choose \( u \) and \( dv \) so that \( u \) becomes “simpler” when differentiated, while leaving a \( dv \) that can be readily integrated to obtain \( v \). Thus, for the integral \( \int x \cos x \, dx \) in Example 1, both goals were achieved by letting \( u = x \) and \( dv = \cos x \, dx \). In contrast, \( u = \cos x \) would not have been a good first choice in that example, since \( du/dx = \sin x \) is no simpler than \( u \). Indeed, had we chosen

\( u = \cos x \quad \text{and} \quad dv = x \, dx \)

then we would have obtained

\[ \int x \cos x \, dx = \frac{x^2}{2} \cos x - \int \frac{x^2}{2} \, (- \sin x) \, dx = \frac{x^2}{2} \cos x + \frac{1}{2} \int x^2 \sin x \, dx \]

For this choice of \( u \) and \( dv \), the new integral is actually more complicated than the original.
There is another useful strategy for choosing $u$ and $dv$ that can be applied when the integrand is a product of two functions from different categories in the list

Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential

In this case you will often be successful if you take $u$ to be the function whose category occurs earlier in the list and take $dv$ to be the rest of the integrand. The acronym LIATE will help you to remember the order. The method does not work all the time, but it works often enough to be useful.

Note, for example, that the integrand in Example 1 consists of the product of the algebraic function $x$ and the trigonometric function $\cos x$. Thus, the LIATE method suggests that we should let $u = x$ and $dv = \cos x\,dx$, which proved to be a successful choice.

Example 2
Evaluate $\int xe^{x}\,dx$.

Solution. In this case the integrand is the product of the algebraic function $x$ with the exponential function $e^x$. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^{x}\,dx$$

so that

$$du = dx \quad \text{and} \quad v = \int e^{x}\,dx = e^{x}$$

Thus, from (3)

$$\int xe^{x}\,dx = \int u\,dv = uv - \int v\,du = x\,e^{x} - \int e^{x}\,dx = xe^{x} - e^{x} + C \triangleleft$$

Example 3
Evaluate $\int \ln x\,dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x\,dx$. But with this choice finding $v$ is equivalent to evaluating $\int \ln x\,dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x}\,dx \quad v = \int dx = x$$

With this choice it follows from (3) that

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C \triangleleft$$

REPEATED INTEGRATION BY PARTS
It is sometimes necessary to use integration by parts more than once in the same problem.

Example 4
Evaluate $\int x^{2}e^{-x}\,dx$.

Solution. Let

$$u = x^{2}, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$
so that from (3)
\[ \int x^2 e^{-x} \, dx = \int u \, dv = uv - \int v \, du \]
\[ = x^2 (-e^{-x}) - \int -e^{-x} (2x) \, dx \]
\[ = -x^2 e^{-x} + 2 \int x e^{-x} \, dx \]  
(4)

The last integral is similar to the original except that we have replaced \( x^2 \) by \( x \). Another integration by parts applied to \( \int x e^{-x} \, dx \) will complete the problem. We let
\[ u = x, \quad dv = e^{-x} \, dx, \quad du = dx, \quad v = \int e^{-x} \, dx = -e^{-x} \]
so that
\[ \int x e^{-x} \, dx = x(-e^{-x}) - \int -e^{-x} \, dx = -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} + C \]

Finally, substituting this into the last line of (4) yields
\[ \int x^2 e^{-x} \, dx = -x^2 e^{-x} + 2 \int x e^{-x} \, dx = -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) + C \]
\[ = - (x^2 + 2x + 2)e^{-x} + C \]  

The LIATE method suggests that integrals of the form
\[ \int e^{ax} \sin bx \, dx \quad \text{and} \quad \int e^{ax} \cos bx \, dx \]
can be evaluated by letting \( u = \sin bx \) or \( u = \cos bx \) and \( dv = e^{ax} \, dx \). However, this will require a technique that deserves special attention.

Example 5 Evaluate \( \int e^x \cos x \, dx \).

Solution. Let
\[ u = \cos x, \quad dv = e^x \, dx, \quad du = -\sin x \, dx, \quad v = \int e^x \, dx = e^x \]
Thus,
\[ \int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du = e^x \cos x + \int e^x \sin x \, dx \]  
(5)

Since the integral \( \int e^x \sin x \, dx \) is similar in form to the original integral \( \int e^x \cos x \, dx \), it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let
\[ u = \sin x, \quad dv = e^x \, dx, \quad du = \cos x \, dx, \quad v = \int e^x \, dx = e^x \]
Thus,
\[ \int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \cos x \, dx \]
Together with Equation (5) this yields
\[ \int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx \]  
(6)
which is an equation we can solve for the unknown integral. We obtain

\[ 2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x \]

and hence

\[ \int e^x \cos x \, dx = \frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C \]

### A Tabular Method for Repeated Integration by Parts

Integrals of the form

\[ \int p(x) f(x) \, dx \]

where \( p(x) \) is a polynomial, can sometimes be evaluated using repeated integration by parts in which \( u \) is taken to be \( p(x) \) or one of its derivatives at each stage. Since \( du \) is computed by differentiating \( u \), the repeated differentiation of \( p(x) \) will eventually produce 0, at which point you may be left with a simplified integration problem. A convenient method for organizing the computations into two columns is called **tabular integration by parts**.

**Tabular Integration by Parts**

**Step 1.** Differentiate \( p(x) \) repeatedly until you obtain 0, and list the results in the first column.

**Step 2.** Integrate \( f(x) \) repeatedly and list the results in the second column.

**Step 3.** Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

**Step 4.** Label the arrows with alternating + and − signs, starting with a +.

**Step 5.** For each arrow, form the product of the expressions at its tip and tail and then multiply that product by +1 or −1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

This process is illustrated in Figure 7.2.1 for the integral \( \int (x^2 - x) \cos x \, dx \).

![Figure 7.2.1](image)

**Example 6** In Example 11 of Section 5.3 we evaluated \( \int \sqrt{x^2 - 1} \, dx \) using \( u \)-substitution. Evaluate this integral using tabular integration by parts.
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Solution.

<table>
<thead>
<tr>
<th>REPEATED DIFFERENTIATION</th>
<th>REPEATED INTEGRATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>$(x - 1)^{3/2}$</td>
</tr>
<tr>
<td>$2x$</td>
<td>$rac{3}{2}(x - 1)^{3/2}$</td>
</tr>
<tr>
<td>$2$</td>
<td>$rac{5}{4}(x - 1)^{5/2}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$rac{8}{105}(x - 1)^{7/2}$</td>
</tr>
</tbody>
</table>

Thus, it follows that

$$\int x^2 \sqrt{x - 1} \, dx = \frac{2}{3} x^2 (x - 1)^{3/2} - \frac{8}{15} x (x - 1)^{5/2} + \frac{16}{105} (x - 1)^{7/2} + C \quad \blacktriangleleft$$

INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

For definite integrals the formula corresponding to (3) is

$$\int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du$$  \hspace{1cm} (7)

Remark It is important to keep in mind that the variables $u$ and $v$ in this formula are functions of $x$ and that the limits of integration in (7) are limits on the variable $x$. Sometimes it is helpful to emphasize this by writing (7) as

$$\int_{x=a}^{x=b} u \, dv = u v \bigg|_{x=a}^{x=b} - \int_{x=a}^{x=b} v \, du$$ \hspace{1cm} (8)

The next example illustrates how integration by parts can be used to integrate the inverse trigonometric functions.

Example 7 Evaluate $\int_0^1 \tan^{-1} x \, dx$.

Solution. Let $u = \tan^{-1} x$, $dv = dx$, $du = \frac{1}{1 + x^2} \, dx$, $v = x$

Thus,

$$\int_0^1 \tan^{-1} x \, dx = \left[ \int_0^1 u \, dv = uv \right]_0^1 - \int_0^1 v \, du$$

$$= x \tan^{-1} x \bigg|_0^1 - \int_0^1 \frac{x}{1 + x^2} \, dx$$

But

$$\int_0^1 \frac{x}{1 + x^2} \, dx = \frac{1}{2} \int_0^1 \frac{2x}{1 + x^2} \, dx = \frac{1}{2} \ln(1 + x^2) \bigg|_0^1 = \frac{1}{2} \ln 2$$

so

$$\int_0^1 \tan^{-1} x \, dx = x \tan^{-1} x \bigg|_0^1 - \frac{1}{2} \ln 2 = (\frac{\pi}{4} - 0) - \frac{1}{2} \ln 2 = \frac{\pi}{4} - \ln \sqrt{2} \quad \blacktriangleleft$$
7.2 Integration by Parts

**REDUCTION FORMULAS**

Integration by parts can be used to derive *reduction formulas* for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a lower power of that function. For example, if \( n \) is a positive integer and \( n \geq 2 \), then integration by parts can be used to obtain the reduction formulas

\[
\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \tag{9}
\]

\[
\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \tag{10}
\]

To illustrate how such formulas can be obtained, let us derive (10). We begin by writing \( \cos^n x \) as \( \cos^{n-1} x \cdot \cos x \) and letting

\[
u = \cos^{n-1} x \quad dv = \cos x \, dx
\]

\[
du = (n-1) \cos^{n-2} x (-\sin x) \, dx \quad v = \sin x
\]

so that

\[
\int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx = \int u \, dv = uv - \int v \, du
\]

\[
= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx
\]

\[
= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx
\]

\[
= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx
\]

Moving the last term on the right to the left side yields

\[
n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx
\]

from which (10) follows. The derivation of reduction formula (9) is similar (Exercise 63).

Reduction formulas (9) and (10) reduce the exponent of sine (or cosine) by 2. Thus, if the formulas are applied repeatedly, the exponent can eventually be reduced to 0 if \( n \) is even or 1 if \( n \) is odd, at which point the integration can be completed. We will discuss this method in more detail in the next section, but for now, here is an example that illustrates how reduction formulas work.

► **Example 8** Evaluate \( \int \cos^4 x \, dx \).

**Solution.** From (10) with \( n = 4 \)

\[
\int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx \tag{Now apply (10) with \( n = 2 \)}
\]

\[
= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left( \frac{1}{2} \cos x \sin x + \frac{1}{2} \int dx \right)
\]

\[
= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C
\]
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Quick check exercises 7.2 (See page 500 for answers.)

1. (a) If \( G'(x) = g(x) \), then
\[
\int f(x)g(x) \, dx = f(x)G(x) - ________
\]
(b) If \( u = f(x) \) and \( v = G(x) \), then the formula in part (a)
can be written in the form \( \int u \, dv = ________ \).

2. Find an appropriate choice of \( u \) and \( dv \) for integration by
parts of each integral. Do not evaluate the integral.
(a) \( \int x \ln x \, dx; \ u = ________, \ dv = ________ \)
(b) \( \int (x - 2) \sin x \, dx; \ u = ________, \ dv = ________ \)

Exercise set 7.2

1–38 Evaluate the integral.
1. \( \int xe^{-2x} \, dx \)
2. \( \int xe^{3x} \, dx \)
3. \( \int xe^{2x} \, dx \)
4. \( \int xe^{-2x} \, dx \)
5. \( \int x \sin 3x \, dx \)
6. \( \int x \cos 2x \, dx \)
7. \( \int x^2 \cos x \, dx \)
8. \( \int x^2 \sin x \, dx \)
9. \( \int x \ln x \, dx \)
10. \( \int \sqrt{x} \ln x \, dx \)
11. \( \int (\ln x)^2 \, dx \)
12. \( \int \frac{\ln x}{\sqrt{x}} \, dx \)
13. \( \int \ln(3x - 2) \, dx \)
14. \( \int \ln(x^2 + 4) \, dx \)
15. \( \int \sin^{-1} x \, dx \)
16. \( \int \cos^{-1}(2x) \, dx \)
17. \( \int \tan^{-1}(3x) \, dx \)
18. \( \int x \tan^{-1} x \, dx \)
19. \( \int e^x \sin x \, dx \)
20. \( \int e^{3x} \cos 2x \, dx \)
21. \( \int \sin(\ln x) \, dx \)
22. \( \int \cos(\ln x) \, dx \)
23. \( \int x \sec^2 x \, dx \)
24. \( \int x \tan^2 x \, dx \)
25. \( \int x^3 e^x \, dx \)
26. \( \int \frac{xe^x}{(x + 1)^2} \, dx \)
27. \( \int_0^1 xe^{2x} \, dx \)
28. \( \int_0^1 xe^{-3x} \, dx \)
29. \( \int_1^\infty x \ln x \, dx \)
30. \( \int_0^\infty \ln x \, dx \)
31. \( \int_1^\infty \ln(x + 2) \, dx \)
32. \( \int_0^{\sqrt{3}/2} \sin^{-1} x \, dx \)
33. \( \int_2^4 \sec^{-1} \sqrt{\theta} \, d\theta \)
34. \( \int_1^2 x \sec^{-1} x \, dx \)
35. \( \int_0^\pi x \sin 2x \, dx \)
36. \( \int_0^\pi (x + x \cos x) \, dx \)
37. \( \int_1^3 \sqrt{x} \tan^{-1} \sqrt{x} \, dx \)
38. \( \int_0^2 \ln(x^2 + 1) \, dx \)

39–42 True–False Determine whether the statement is true or false. Explain your answer.
39. The main goal in integration by parts is to choose \( u \) and \( dv \) to obtain a new integral that is easier to evaluate than the original.
40. Applying the LIATE strategy to evaluate \( \int x^3 \ln x \, dx \), we should choose \( u = x^3 \) and \( dv = \ln x \, dx \).
41. To evaluate \( \int \ln e^x \, dx \) using integration by parts, choose \( dv = \ln x \, dx \).
42. Tabular integration by parts is useful for integrals of the form \( \int p(x)f(x) \, dx \), where \( p(x) \) is a polynomial and \( f(x) \) can be repeatedly integrated.

43–44 Evaluate the integral by making a \( u \)-substitution and then integrating by parts.
43. \( \int e^{\sqrt{x}} \, dx \)
44. \( \int \cos \sqrt{x} \, dx \)
45. Prove that tabular integration by parts gives the correct answer for
\[
\int p(x)f(x) \, dx
\]
where \( p(x) \) is any quadratic polynomial and \( f(x) \) is any function that can be repeatedly integrated.
46. The computations of any integral evaluated by repeated integration by parts can be organized using tabular integration by parts. Use this organization to evaluate \( \int e^x \cos x \, dx \) in
two ways: first by repeated differentiation of $\cos x$ (compare Example 5), and then by repeated differentiation of $e^x$.

**47–52** Evaluate the integral using tabular integration by parts.

47. $\int (3x^2 - x + 2)e^{-x} \, dx$
48. $\int (x^2 + x + 1) \sin x \, dx$
49. $\int 4x^4 \sin 2x \, dx$
50. $\int x^3 \sqrt{2x + 1} \, dx$
51. $\int e^{3x} \sin bx \, dx$
52. $\int e^{-30} \sin 5\theta \, d\theta$

**53.** Consider the integral $\int \sin x \cos x \, dx$.
(a) Evaluate the integral two ways: first using integration by parts, and then using the substitution $u = \sin x$.
(b) Show that the results of part (a) are equivalent.
(c) Which of the two methods do you prefer? Discuss the reasons for your preference.

54. Evaluate the integral
$$\int_0^1 \frac{x^3}{\sqrt{x^2 + 1}} \, dx$$
using
(a) integration by parts
(b) the substitution $u = \sqrt{x^2 + 1}$.

**55.** (a) Find the area of the region enclosed by $y = \ln x$, the line $x = e$, and the $x$-axis.
(b) Find the volume of the solid generated when the region in part (a) is revolved about the $x$-axis.

56. Find the area of the region between $y = x \sin x$ and $y = x$ for $0 \leq x \leq \pi/2$.

57. Find the volume of the solid generated when the region between $y = \sin x$ and $y = 0$ for $0 \leq x \leq \pi$ is revolved about the $y$-axis.

58. Find the volume of the solid generated when the region enclosed by $y = \cos x$ and $y = 0$ for $0 \leq x \leq \pi/2$ is revolved about the $y$-axis.

59. A particle moving along the $x$-axis has velocity function $v(t) = t^2 \sin t$. How far does the particle travel from time $t = 0$ to $t = \pi$?

60. The study of sawtooth waves in electrical engineering leads to integrals of the form
$$\int_{-\pi/\omega}^{\pi/\omega} t \sin(kt \sin t) \, dt$$
where $k$ is an integer and $\omega$ is a nonzero constant. Evaluate the integral.

61. Use reduction formula (9) to evaluate
(a) $\int \sin^3 x \, dx$
(b) $\int_0^{\pi/2} \sin^5 x \, dx$.

62. Use reduction formula (10) to evaluate
(a) $\int \cos^3 x \, dx$
(b) $\int_0^{\pi/2} \cos^6 x \, dx$.

63. Derive reduction formula (9).

**7.2 Integration by Parts**

64. In each part, use integration by parts or other methods to derive the reduction formula.
(a) $\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$
(b) $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} + \int \tan^{n-2} x \, dx$
(c) $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$

65–66 Use the reduction formulas in Exercise 64 to evaluate the integrals.

65. (a) $\int \tan^4 x \, dx$
(b) $\int \sec^4 x \, dx$
(c) $\int x^3 e^x \, dx$

66. (a) $\int x^2 e^{3x} \, dx$
(b) $\int_0^1 x e^{-x^2} \, dx$  
[Hint: First make a substitution.]

67. Let $f$ be a function whose second derivative is continuous on $[-1, 1]$. Show that
$$\int_{-1}^1 x f''(x) \, dx = f'(1) + f'(-1) - f(1) - f(-1)$$

**FOCUS ON CONCEPTS**

68. (a) In the integral $\int x \cos x \, dx$, let $u = x$, $dv = \cos x \, dx$,
$du = dx$,  $v = \sin x + C$
Show that the constant $C$ cancels out, thus giving the same solution obtained by omitting $C$.
(b) Show that in general
$$uv - \int v \, du = u(v + C) - \int (v + C) \, du$$
thereby justifying the omission of the constant of integration when calculating $v$ in integration by parts.

69. Evaluate $\int \ln(x + 1) \, dx$ using integration by parts. Simplify the computation of $\int v \, du$ by introducing a constant of integration $C_1 = 1$ when going from $dv$ to $v$.

70. Evaluate $\int \ln(3x - 2) \, dx$ using integration by parts. Simplify the computation of $\int v \, du$ by introducing a constant of integration $C_1 = -\frac{3}{2}$ when going from $dv$ to $v$. Compare your solution with your answer to Exercise 13.

71. Evaluate $\int x \tan^{-1} x \, dx$ using integration by parts. Simplify the computation of $\int v \, du$ by introducing a constant of integration $C_1 = \frac{1}{2}$ when going from $dv$ to $v$.

72. What equation results if integration by parts is applied to the integral $\int \frac{1}{x \ln x} \, dx$ with the choices $u = \frac{1}{\ln x}$ and $dv = \frac{1}{x} \, dx$?
In what sense is this equation true? In what sense is it false?
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73. Writing Explain how the product rule for derivatives and the technique of integration by parts are related.

74. Writing For what sort of problems are the integration techniques of substitution and integration by parts “competing” techniques? Describe situations, with examples, where each of these techniques would be preferred over the other.

Quick Check Answers 7.2

1. (a) \( \int f'(x)G(x)\,dx \) (b) \( uv - \int v\,du \) 2. (a) \( \ln x \); \( x\,dx \) (b) \( x - 2 \); \( \sin x\,dx \) (c) \( \sin^{-1}x\,dx \) (d) \( x; \frac{1}{\sqrt{x-1}}\,dx \)

3. (a) \( \left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + C \) (b) \( (x - 1)\ln(x - 1) - x + C \) (c) \( \frac{1}{x} \) 4. \( -\frac{1}{3}\sin^2 x \cos x - \frac{1}{2}\cos x + C \)

Integrating Trigonometric Functions

In the last section we derived reduction formulas for integrating positive integer powers of sine, cosine, tangent, and secant. In this section we will show how to work with those reduction formulas, and we will discuss methods for integrating other kinds of integrals that involve trigonometric functions.

Integrating Powers of Sine and Cosine

We begin by recalling two reduction formulas from the preceding section.

\[
\int \sin^n x\,dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\,dx \quad (1)
\]

\[
\int \cos^n x\,dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\,dx \quad (2)
\]

In the case where \( n = 2 \), these formulas yield

\[
\int \sin^2 x\,dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx = \frac{1}{2}x - \frac{1}{2} \sin x \cos x + C \quad (3)
\]

\[
\int \cos^2 x\,dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} \int dx = \frac{1}{2}x + \frac{1}{2} \sin x \cos x + C \quad (4)
\]

Alternative forms of these integration formulas can be derived from the trigonometric identities

\[
\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad (5–6)
\]

which follow from the double-angle formulas

\[
\cos 2x = 1 - 2\sin^2 x \quad \text{and} \quad \cos 2x = 2\cos^2 x - 1
\]

These identities yield

\[
\int \sin^2 x\,dx = \frac{1}{2} \int (1 - \cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4} \sin 2x + C \quad (7)
\]

\[
\int \cos^2 x\,dx = \frac{1}{2} \int (1 + \cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4} \sin 2x + C \quad (8)
\]
7.3 Integrating Trigonometric Functions

Observe that the antiderivatives in Formulas (3) and (4) involve both sines and cosines, whereas those in (7) and (8) involve sines alone. However, the apparent discrepancy is easy to resolve by using the identity

\[ \sin 2x = 2 \sin x \cos x \]

to rewrite (7) and (8) in forms (3) and (4), or conversely.

In the case where \( n = 3 \), the reduction formulas for integrating \( \sin^3 x \) and \( \cos^3 x \) yield

\[
\begin{align*}
\int \sin^3 x \, dx &= -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx = -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \cos x + C \\
\int \cos^3 x \, dx &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \int \cos x \, dx = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C
\end{align*}
\]

(9) (10)

If desired, Formula (9) can be expressed in terms of cosines alone by using the identity \( \sin^2 x = 1 - \cos^2 x \), and Formula (10) can be expressed in terms of sines alone by using the identity \( \cos^2 x = 1 - \sin^2 x \). We leave it for you to do this and confirm that

\[
\begin{align*}
\int \sin^3 x \, dx &= \frac{1}{4} \cos^3 x - \cos x + C \\
\int \cos^3 x \, dx &= \sin x - \frac{1}{4} \sin^3 x + C
\end{align*}
\]

(11) (12)

We leave it as an exercise to obtain the following formulas by first applying the reduction formulas, and then using appropriate trigonometric identities.

\[
\begin{align*}
\int \sin^4 x \, dx &= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C \\
\int \cos^4 x \, dx &= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C
\end{align*}
\]

(13) (14)

Example 1  Find the volume \( V \) of the solid that is obtained when the region under the curve \( y = \sin^2 x \) over the interval \([0, \pi]\) is revolved about the \( x \)-axis (Figure 7.3.1).

Solution.  Using the method of disks, Formula (5) of Section 6.2, and Formula (13) above yields

\[
V = \int_0^\pi \pi \sin^4 x \, dx = \pi \left[ \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x \right]_0^{\pi} = \frac{3}{8} \pi^2
\]

Example 2  Evaluate

(a) \( \int \sin^4 x \cos^3 x \, dx \)  (b) \( \int \sin^4 x \cos^4 x \, dx \)
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Table 7.3.1

<table>
<thead>
<tr>
<th>( \int \sin^m x \cos^n x , dx )</th>
<th>PROCEDURE</th>
<th>RELEVANT IDENTITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m ) odd</td>
<td>• Split off a factor of ( \cos x ).</td>
<td>( \cos^2 x = 1 - \sin^2 x )</td>
</tr>
<tr>
<td></td>
<td>• Apply the relevant identity.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Make the substitution ( u = \sin x ).</td>
<td></td>
</tr>
<tr>
<td>( n ) odd</td>
<td>• Split off a factor of ( \sin x ).</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Apply the relevant identity.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Make the substitution ( u = \cos x ).</td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
\int \sin^m x \cos^n x \, dx &= \int \sin^m x (1 - \sin^2 x)^{n/2} \, dx \\
&= \int u^m (1 - u^2)^{n/2} \, du \\
&= \int (u^2 - 2u^6 + u^8) \, du \\
&= \frac{1}{3}u^3 - \frac{1}{7}u^7 + \frac{1}{9}u^9 + C \\
&= \frac{1}{3}\sin^3 x - \frac{1}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C
\end{align*} \]

Solution (a). Since \( n = 5 \) is odd, we will follow the first procedure in Table 7.3.1:

\[ \begin{align*}
\int \sin^5 x \cos^4 x \, dx &= \int \sin^4 x \cos^5 x \, dx \\
&= \int \sin^4 x (1 - \sin^2 x)^3 \cos x \, dx \\
&= \int u^4 (1 - u^2)^3 \, du \\
&= \int (u^6 - 2u^8 + u^{10}) \, du \\
&= \frac{1}{7}u^7 - \frac{2}{9}u^9 + \frac{1}{11}u^{11} + C \\
&= \frac{1}{7}\sin^7 x - \frac{2}{9}\sin^9 x + \frac{1}{11}\sin^{11} x + C
\end{align*} \]

Solution (b). Since \( m = n = 4 \), both exponents are even, so we will follow the third procedure in Table 7.3.1:

\[ \begin{align*}
\int \sin^4 x \cos^4 x \, dx &= \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx \\
&= \int \left( \frac{1}{2} (1 - \cos 2x) \right)^2 (\frac{1}{2} (1 + \cos 2x))^2 \, dx \\
&= \frac{1}{16} \int (1 - \cos 2x)^2 \, dx \\
&= \frac{1}{16} \int \sin^4 2x \, dx \\
&= \frac{1}{16} \int \sin^4 u \, du \\
&= \frac{1}{16} \int \sin^4 u \, du \quad \text{or} \quad \frac{1}{16} \int \sin 4u \, du \\
&= \frac{1}{32} \left( \frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u \right) + C \\
&= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C
\end{align*} \]

Note that this can be obtained more directly from the original integral using the identity \( \sin x \cos x = \frac{1}{2} \sin 2x \).
7.3 Integrating Trigonometric Functions

Integrals of the form
\[ \int \sin mx \cos nx \, dx, \quad \int \sin mx \sin nx \, dx, \quad \int \cos mx \cos nx \, dx \] (15)
can be found by using the trigonometric identities
\[ \sin \alpha \cos \beta = \frac{1}{2} [\sin (\alpha - \beta) + \sin (\alpha + \beta)] \] (16)
\[ \sin \alpha \sin \beta = \frac{1}{2} [\cos (\alpha - \beta) - \cos (\alpha + \beta)] \] (17)
\[ \cos \alpha \cos \beta = \frac{1}{2} [\cos (\alpha - \beta) + \cos (\alpha + \beta)] \] (18)
to express the integrand as a sum or difference of sines and cosines.

Example 3
Evaluate \( \int \sin 7x \cos 3x \, dx \).

Solution. Using (16) yields
\[ \int \sin 7x \cos 3x \, dx = \frac{1}{2} \int (\sin 4x + \sin 10x) \, dx = -\frac{1}{8} \cos 4x - \frac{1}{20} \cos 10x + C \]

INTEGRATING POWERS OF TANGENT AND SECANT

The procedures for integrating powers of tangent and secant closely parallel those for sine and cosine. The idea is to use the following reduction formulas (which were derived in Exercise 64 of Section 7.2) to reduce the exponent in the integrand until the resulting integral can be evaluated:

\[ \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \] (19)
\[ \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \] (20)

In the case where \( n \) is odd, the exponent can be reduced to 1, leaving us with the problem of integrating \( \tan x \) or \( \sec x \). These integrals are given by

\[ \int \tan x \, dx = \ln |\sec x| + C \] (21)
\[ \int \sec x \, dx = \ln |\sec x + \tan x| + C \] (22)

Formula (21) can be obtained by writing
\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln |\cos x| + C \quad \text{or} \quad u = \cos x, \quad du = -\sin x \, dx \]
\[ = \ln |\sec x| + C \quad \text{or} \quad \ln \left| \cos x \right| = -\ln \left| \cos x \right| \]

To obtain formula (22) we write
\[ \int \sec x \, dx = \int \sec x \left( \frac{\sec x + \tan x}{\sec x + \tan x} \right) \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \]
\[ = \ln |\sec x + \tan x| + C \quad \text{or} \quad u = \sec x + \tan x, \quad du = (\sec^2 x + \sec x \tan x) \, dx \]
The following basic integrals occur frequently and are worth noting:

\[
\int \tan^2 x \, dx = \tan x - x + C \quad (23)
\]

\[
\int \sec^2 x \, dx = \tan x + C \quad (24)
\]

Formula (24) is already known to us, since the derivative of \( \tan x \) is \( \sec^2 x \). Formula (23) can be obtained by applying reduction formula (19) with \( n = 2 \) (verify) or, alternatively, by using the identity

\[
1 + \tan^2 x = \sec^2 x
\]

to write

\[
\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C
\]

The formulas

\[
\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \ln |\sec x| + C \quad (25)
\]

\[
\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C \quad (26)
\]

can be deduced from (21), (22), and reduction formulas (19) and (20) as follows:

\[
\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx = \frac{1}{2} \tan^2 x - \ln |\sec x| + C
\]

\[
\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C
\]

### INTEGRATING PRODUCTS OF TANGENTS AND SECANTS

If \( m \) and \( n \) are positive integers, then the integral

\[
\int \tan^m x \sec^n x \, dx
\]

can be evaluated by one of the three procedures stated in Table 7.3.2, depending on whether \( m \) and \( n \) are odd or even.

**Table 7.3.2**

<table>
<thead>
<tr>
<th>( \int \tan^m x \sec^n x , dx )</th>
<th>PROCEDURE</th>
<th>RELEVANT IDENTITIES</th>
</tr>
</thead>
</table>
| \( n \) even                      | • Split off a factor of \( \sec^2 x \).
|                                    | • Apply the relevant identity. |
|                                    | • Make the substitution \( u = \tan x \). |
|                                    | sec^2 x = \tan^2 x + 1 |
| \( m \) odd                       | • Split off a factor of \( \sec x \tan x \).
|                                    | • Apply the relevant identity. |
|                                    | • Make the substitution \( u = \sec x \). |
|                                    | tan^2 x = \sec^2 x - 1 |
| \( m \) even \& \( n \) odd      | • Use the relevant identities to reduce the integrand to powers of \( \sec x \) alone. |
|                                    | • Then use the reduction formula for powers of \( \sec x \). |
|                                    | tan^2 x = \sec^2 x - 1 |
Example 4  Evaluate

(a)  \( \int \tan^2 x \sec^4 x \, dx \)  (b)  \( \int \tan^3 x \sec^3 x \, dx \)  (c)  \( \int \tan^2 x \sec x \, dx \)

Solution (a).  Since \( n = 4 \) is even, we will follow the first procedure in Table 7.3.2:

\[
\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \sec^2 x \, dx
\]
\[
= \int \tan^2 x \sec^2 x (\tan^2 x + 1) \, dx
\]
\[
= \int u^2 (u^2 + 1) \, du
\]
\[
= \frac{1}{5} u^5 + \frac{1}{3} u^3 + C = \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C
\]

Solution (b).  Since \( m = 3 \) is odd, we will follow the second procedure in Table 7.3.2:

\[
\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x (\sec x \tan x) \, dx
\]
\[
= \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x) \, dx
\]
\[
= \int (a^2 - 1)u^2 \, du
\]
\[
= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C
\]

Solution (c).  Since \( m = 2 \) is even and \( n = 1 \) is odd, we will follow the third procedure in Table 7.3.2:

\[
\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx
\]
\[
= \int \sec^3 x \, dx - \int \sec x \, dx
\]
\[
= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| - \ln |\sec x + \tan x| + C
\]
\[
= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C
\]

AN ALTERNATIVE METHOD FOR INTEGRATING POWERS OF SINE, COSINE, TANGENT, AND SECANT

The methods in Tables 7.3.1 and 7.3.2 can sometimes be applied if \( m = 0 \) or \( n = 0 \) to integrate positive integer powers of sine, cosine, tangent, and secant without reduction formulas. For example, instead of using the reduction formula to integrate \( \sin^3 x \), we can apply the second procedure in Table 7.3.1:

\[
\int \sin^3 x \, dx = \int (\sin^2 x) \sin x \, dx
\]
\[
= \int (1 - \cos^2 x) \sin x \, dx
\]
\[
= \int (1 - u^2) \, du
\]
\[
= \frac{1}{3} u^3 - u + C = \frac{1}{3} \cos^3 x - \cos x + C
\]

which agrees with (11).
MERCATOR’S MAP OF THE WORLD

The integral of \( \sec x \) plays an important role in the design of navigational maps for charting nautical and aeronautical courses. Sailors and pilots usually chart their courses along paths with constant compass headings; for example, the course might be 30° northeast or 135° southeast. Except for courses that are parallel to the equator or run due north or south, a course with constant compass heading spirals around the Earth toward one of the poles (as in the top part of Figure 7.3.2). In 1569 the Flemish mathematician and geographer Gerhard Kramer (1512–1594) (better known by the Latin name Mercator) devised a world map, called the Mercator projection, in which spirals of constant compass headings appear as straight lines. This was extremely important because it enabled sailors to determine compass headings between two points by connecting them with a straight line on a map (as in the bottom part of Figure 7.3.2).

If the Earth is assumed to be a sphere of radius 4000 mi, then the lines of latitude at 1° increments are equally spaced about 70 mi apart (why?). However, in the Mercator projection, the lines of latitude become wider apart toward the poles, so that two widely spaced latitude lines near the poles may be actually the same distance apart on the Earth as two closely spaced latitude lines near the equator. It can be proved that on a Mercator map in which the equatorial line has length \( L \), the vertical distance \( D_\beta \) on the map between the equator (latitude 0°) and the line of latitude \( \beta \)° is

\[
D_\beta = \frac{L}{2\pi} \int_{0}^{\beta \pi/180} \sec x \, dx
\]

(27)

QUICK CHECK EXERCISES 7.3

(See page 508 for answers.)

1. Complete each trigonometric identity with an expression involving \( \cos 2x \).
   (a) \( \sin^2 x = \) ________
   (b) \( \cos^2 x = \) ________
   (c) \( \cos^2 x - \sin^2 x = \) ________

2. Evaluate the integral.
   (a) \( \int \sec^2 x \, dx = \) ________
   (b) \( \int \tan^2 x \, dx = \) ________
   (c) \( \int \sec x \, dx = \) ________
   (d) \( \int \tan x \, dx = \) ________

3. Use the indicated substitution to rewrite the integral in terms of \( u \). Do not evaluate the integral.
   (a) \( \int \sin^2 x \cos x \, dx; \ u = \sin x \)
   (b) \( \int \sin^3 x \cos^2 x \, dx; \ u = \cos x \)
   (c) \( \int \tan^3 x \sec^2 x \, dx; \ u = \tan x \)
   (d) \( \int \tan^3 x \sec x \, dx; \ u = \sec x \)

EXERCISE SET 7.3

1–52 Evaluate the integral.

1. \( \int \cos^3 x \sin x \, dx \)
2. \( \int \sin^3 3x \cos 3x \, dx \)
3. \( \int \sin^2 5\theta \, d\theta \)
4. \( \int \cos^2 3x \, dx \)
5. \( \int \sin^3 a\theta \, d\theta \)
6. \( \int \cos^3 at \, dt \)
7. \( \int \sin ax \cos ax \, dx \)
8. \( \int \sin^3 x \cos^3 x \, dx \)
9. \( \int \sin^2 t \cos^2 t \, dt \)
10. \( \int \sin^3 x \cos^2 x \, dx \)
53–56 True-False Determine whether the statement is true or false. Explain your answer.

53. To evaluate \( \int \sin^2 x \cos^2 x \, dx \), use the trigonometric identity \( \sin^2 x = 1 - \cos^2 x \) and the substitution \( u = \cos x \).

54. To evaluate \( \int \sin^2 x \cos^2 x \, dx \), use the trigonometric identity \( \sin^2 x = 1 - \cos^2 x \) and the substitution \( u = \cos x \).

55. The trigonometric identity
\[
\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]
\]
is often useful for evaluating integrals of the form \( \int \sin^m x \cos^n x \, dx \).

56. The integral \( \int \tan^4 x \sec^5 x \, dx \) is equivalent to one whose integrand is a polynomial in \( \sec x \).

57. Let \( m, n \) be distinct nonnegative integers. Use Formulas (16)–(18) to prove:
(a) \( \int_0^{2\pi} \sin mx \cos nx \, dx = 0 \)
(b) \( \int_0^{2\pi} \cos mx \cos nx \, dx = 0 \)
(c) \( \int_0^{2\pi} \sin mx \sin nx \, dx = 0 \).

58. Evaluate the integrals in Exercise 57 when \( m \) and \( n \) denote the same nonnegative integer.

59. Find the arc length of the curve \( y = \ln(\cos x) \) over the interval \([0, \pi/4]\).

60. Find the volume of the solid generated when the region enclosed by \( y = \tan x \), \( y = 1 \), and \( x = 0 \) is revolved about the \( x \)-axis.

61. Find the volume of the solid that results when the region enclosed by \( y = \cos x \), \( y = \sin x \), \( x = 0 \), and \( x = \pi/4 \) is revolved about the \( x \)-axis.

62. The region bounded below by the \( x \)-axis and above by the portion of \( y = \sin x \) from \( x = 0 \) to \( x = \pi \) is revolved about the \( x \)-axis. Find the volume of the resulting solid.

63. Use Formula (27) to show that if the length of the equatorial line on a Mercator projection is \( L \), then the vertical distance \( D \) between the latitude lines at \( \alpha^\circ \) and \( \beta^\circ \) on the same side of the equator (where \( \alpha < \beta \)) is
\[
D = \frac{L}{2\pi} \ln \left( \frac{\sec \beta^\circ + \tan \beta^\circ}{\sec \alpha^\circ + \tan \alpha^\circ} \right)
\]

64. Suppose that the equator has a length of 100 cm on a Mercator projection. In each part, use the result in Exercise 63 to answer the question.
(a) What is the vertical distance on the map between the equator and the line at \( 25^\circ \) north latitude?
(b) What is the vertical distance on the map between New Orleans, Louisiana, at \( 30^\circ \) north latitude and Winnipeg, Canada, at \( 50^\circ \) north latitude?

**FOCUS ON CONCEPTS**

66. (a) Show that
\[
\int \csc x \, dx = -\ln |\csc x + \cot x| + C
\]
(b) Show that the result in part (a) can also be written as
\[
\int \csc x \, dx = \ln |\csc x - \cot x| + C
\]
and
\[
\int \csc x \, dx = \ln |\tan \frac{x}{2}| + C
\]
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66. Rewrite \( \sin x + \cos x \) in the form
\[
A \sin(x + \phi)
\]
and use your result together with Exercise 65 to evaluate
\[
\int \frac{dx}{\sin x + \cos x}
\]

67. Use the method of Exercise 66 to evaluate
\[
\int \frac{dx}{a \sin x + b \cos x} \quad (a, b \text{ not both zero})
\]

68. (a) Use Formula (9) in Section 7.2 to show that
\[
\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \quad (n \geq 2)
\]
(b) Use this result to derive the Wallis sine formulas:
\[
\int_0^{\pi/2} \sin^n x \, dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \quad (n \text{ even} \quad \text{and } \geq 2)
\]
\[
\int_0^{\pi/2} \sin^n x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n} \quad (n \text{ odd} \quad \text{and } \geq 3)
\]

69. Use the Wallis formulas in Exercise 68 to evaluate
\[
\int_0^{\pi/2} \sin^3 x \, dx \quad \int_0^{\pi/2} \sin^4 x \, dx
\]

70. Use Formula (10) in Section 7.2 and the method of Exercise 68 to derive the Wallis cosine formulas:
\[
\int_0^{\pi/2} \cos^n x \, dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \quad (n \text{ even} \quad \text{and } \geq 2)
\]
\[
\int_0^{\pi/2} \cos^n x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n} \quad (n \text{ odd} \quad \text{and } \geq 3)
\]

71. Writing Describe the various approaches for evaluating integrals of the form
\[
\int \sin^m x \cos^n x \, dx
\]
Into what cases do these types of integrals fall? What procedures and identities are used in each case?

72. Writing Describe the various approaches for evaluating integrals of the form
\[
\int \tan^m x \sec^n x \, dx
\]
Into what cases do these types of integrals fall? What procedures and identities are used in each case?

Quick Check Answers 7.3

1. (a) \( \frac{1 - \cos 2x}{2} \) (b) \( \frac{1 + \cos 2x}{2} \) (c) \( \cos 2x \)
2. (a) \( \tan x + C \) (b) \( \tan x - x + C \) (c) \( \ln |\sec x + \tan x| + C \) (d) \( \ln |\sec x| + C \)
3. (a) \( \int u^2 \, du \) (b) \( \int (u^2 - 1)u^2 \, du \) (c) \( \int u^3 \, du \) (d) \( \int (u^2 - 1) \, du \)

7.4 Trigonometric Substitutions

In this section we will discuss a method for evaluating integrals containing radicals by making substitutions involving trigonometric functions. We will also show how integrals containing quadratic polynomials can sometimes be evaluated by completing the square.

The Method of Trigonometric Substitution

To start, we will be concerned with integrals that contain expressions of the form
\[
\sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \sqrt{x^2 - a^2}
\]
in which \( a \) is a positive constant. The basic idea for evaluating such integrals is to make a substitution for \( x \) that will eliminate the radical. For example, to eliminate the radical in the expression \( \sqrt{a^2 - x^2} \), we can make the substitution
\[
x = a \sin \theta, \quad -\pi/2 \leq \theta \leq \pi/2
\]
which yields
\[
\sqrt{a^2 - x^2} = a \sqrt{a^2 - a^2 \sin^2 \theta} = a \sqrt{1 - \sin^2 \theta} = a \cos \theta
\]
\[
= a \cos \theta \quad \text{since } -\pi/2 \leq \theta \leq \pi/2
\]
The restriction on \( \theta \) in (1) serves two purposes—it enables us to replace \(|\cos \theta|\) by \(\cos \theta\) to simplify the calculations, and it also ensures that the substitutions can be rewritten as \(\theta = \sin^{-1}(x/a)\), if needed.

**Example 1**
Evaluate \(\int \frac{dx}{x^2\sqrt{4-x^2}}\).

**Solution.** To eliminate the radical we make the substitution
\[x = 2 \sin \theta, \quad dx = 2 \cos \theta \, d\theta\]
This yields
\[\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2 \cos \theta \, d\theta}{(2 \sin \theta)^2 \sqrt{4-4 \sin^2 \theta}} = \int \frac{2 \cos \theta \, d\theta}{2 \sin \theta \cdot 2 \cos \theta} = \frac{1}{4} \int \frac{d\theta}{\sin^2 \theta} = \frac{1}{4} \int \csc^2 \theta \, d\theta = -\frac{1}{4} \cot \theta + C\]
(2)
At this point we have completed the integration; however, because the original integral was expressed in terms of \(x\), it is desirable to express \(\cot \theta\) in terms of \(x\) as well. This can be done using trigonometric identities, but the expression can also be obtained by writing the substitution \(x = 2 \sin \theta\) as \(\sin \theta = x/2\) and representing it geometrically as in Figure 7.4.1.

From that figure we obtain
\[\cot \theta = \frac{\sqrt{4-x^2}}{x}\]
Substituting this in (2) yields
\[\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4} \frac{\sqrt{4-x^2}}{x} + C\]

**Example 2**
Evaluate \(\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}\).

**Solution.** There are two possible approaches: we can make the substitution in the indefinite integral (as in Example 1) and then evaluate the definite integral using the \(x\)-limits of integration, or we can make the substitution in the definite integral and convert the \(x\)-limits to the corresponding \(\theta\)-limits.

**Method 1.**
Using the result from Example 1 with the \(x\)-limits of integration yields
\[\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4} \left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = -\frac{1}{4} \left[1 - \sqrt{3}\right] = \frac{\sqrt{3} - 1}{4}\]

**Method 2.**
The substitution \(x = 2 \sin \theta\) can be expressed as \(x/2 = \sin \theta\) or \(\theta = \sin^{-1}(x/2)\), so the \(\theta\)-limits that correspond to \(x = 1\) and \(x = \sqrt{2}\) are
\[x = 1: \quad \theta = \sin^{-1}(1/2) = \pi/6\]
\[x = \sqrt{2}: \quad \theta = \sin^{-1}(\sqrt{2}/2) = \pi/4\]
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Thus, from (2) in Example 1 we obtain
\[
\int_1^\sqrt{2} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{1}{4} \int_{\pi/6}^{\pi/4} \csc^2 \theta \, d\theta \quad \text{Convert x-limits to } \theta\text{-limits.}
\]
\[
= -\frac{1}{4} \left[ \cot \theta \right]_{\pi/6}^{\pi/4} = -\frac{1}{4} \left[ 1 - \sqrt{3} \right] = \frac{\sqrt{3} - 1}{4} \quad \blacksquare
\]

Example 3  
Find the area of the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Solution. Because the ellipse is symmetric about both axes, its area \(A\) is four times the area in the first quadrant (Figure 7.4.2). If we solve the equation of the ellipse for \(y\) in terms of \(x\), we obtain
\[
y = \pm \frac{b}{a} \sqrt{a^2 - x^2}
\]
where the positive square root gives the equation of the upper half. Thus, the area \(A\) is given by
\[
A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} \, dx
\]
To evaluate this integral, we will make the substitution \(x = a \sin \theta\) (so \(dx = a \cos \theta \, d\theta\)) and convert the \(x\)-limits of integration to \(\theta\)-limits. Since the substitution can be expressed as \(\theta = \sin^{-1}(x/a)\), the \(\theta\)-limits of integration are
\[
x = 0: \quad \theta = \sin^{-1}(0) = 0
\]
\[
x = a: \quad \theta = \sin^{-1}(1) = \pi/2
\]
Thus, we obtain
\[
A = \frac{4b}{a} \int_0^{\pi/2} a \cos \theta \cdot a \cos \theta \, d\theta
\]
\[
= \frac{4b}{a} \int_0^{\pi/2} \cos^2 \theta \, d\theta = 4ab \int_0^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) \, d\theta
\]
\[
= 2ab \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = 2ab \left[ \frac{\pi}{2} - 0 \right] = \pi ab \quad \blacksquare
\]

REMARK

In the special case where \(a = b\), the ellipse becomes a circle of radius \(a\), and the area formula becomes \(A = \pi a^2\), as expected. It is worth noting that
\[
\int_{-a}^a \sqrt{a^2 - x^2} \, dx = \frac{1}{2} \pi a^2
\]
since this integral represents the area of the upper semicircle (Figure 7.4.3).

Thus far, we have focused on using the substitution \(x = a \sin \theta\) to evaluate integrals involving radicals of the form \(\sqrt{a^2 - x^2}\). Table 7.4.1 summarizes this method and describes some other substitutions of this type.

Technology Mastery

If you have a calculating utility with a numerical integration capability, use it and Formula (3) to approximate \(\pi\) to three decimal places.

Figure 7.4.2

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Figure 7.4.3
7.4 Trigonometric Substitutions

<table>
<thead>
<tr>
<th>Table 7.4.1</th>
<th>TRIGONOMETRIC SUBSTITUTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXPRESSION IN THE INTEGRAND</td>
<td>SUBSTITUTION</td>
</tr>
<tr>
<td>√(a² − x²)</td>
<td>x = a sin θ</td>
</tr>
<tr>
<td>√(a² + x²)</td>
<td>x = a tan θ</td>
</tr>
</tbody>
</table>
| √(x² − a²) | x = a sec θ | 0 ≤ θ < π/2 (if x ≥ a) | \[ x² − a² = a² sec² θ − a² = a² tan² θ \]
|                    |               | π/2 < θ ≤ π (if x ≤ −a) |

Example 4  Find the arc length of the curve \( y = x^2/2 \) from \( x = 0 \) to \( x = 1 \) (Figure 7.4.4).

Solution. From Formula (4) of Section 6.4 the arc length \( L \) of the curve is

\[
L = \int_0^1 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^1 \sqrt{1 + x^2} \, dx
\]

The integrand involves a radical of the form \( \sqrt{a^2 + x^2} \) with \( a = 1 \), so from Table 7.4.1 we make the substitution

\[
x = \tan θ, \quad −π/2 < θ < π/2
\]

\[
\frac{dx}{dθ} = \sec^2 θ \quad \text{or} \quad dx = \sec^2 θ \, dθ
\]

Since this substitution can be expressed as \( θ = \tan^{-1} x \), the \( θ \)-limits of integration that correspond to the \( x \)-limits, \( x = 0 \) and \( x = 1 \), are

\[
x = 0: \quad θ = \tan^{-1} 0 = 0
\]

\[
x = 1: \quad θ = \tan^{-1} 1 = π/4
\]

Thus,

\[
L = \int_0^1 \sqrt{1 + x^2} \, dx = \int_0^{π/4} \sqrt{1 + \tan^2 θ} \sec^2 θ \, dθ
\]

\[
= \int_0^{π/4} \sqrt{\sec^2 θ} \sec^2 θ \, dθ \quad \text{or} \quad 1 + \tan^2 θ = \sec^2 θ
\]

\[
= \int_0^{π/4} \sec θ \sec^2 θ \, dθ
\]

\[
= \int_0^{π/4} \sec^3 θ \, dθ \quad \text{sec} θ > 0 \text{ since } −π/2 < θ < π/2
\]

\[
= \left[ \frac{1}{2} \sec θ \tan θ + \frac{1}{2} \ln |\sec θ + \tan θ| \right]_0^{π/4}
\]

\[
= \left[ \frac{1}{2} \sqrt{2} + \ln(\sqrt{2} + 1) \right] \approx 1.148
\]

Example 5  Evaluate \( \int \frac{\sqrt{x^2 − 25}}{x} \, dx \), assuming that \( x ≥ 5 \).
Solution. The integrand involves a radical of the form $\sqrt{x^2 - a^2}$ with $a = 5$, so from Table 7.4.1 we make the substitution

$$x = 5 \sec \theta, \quad 0 \leq \theta < \pi/2$$

$$dx = 5 \sec \theta \tan \theta \, d\theta \quad \text{or} \quad dx = 5 \sec \theta \tan \theta \, d\theta$$

Thus,

$$\int \frac{\sqrt{x^2 - 25}}{x} \, dx = \int \frac{\sqrt{25 \sec^2 \theta - 25}}{5 \sec \theta} (5 \sec \theta \tan \theta) \, d\theta$$

$$= \int \frac{5 |\tan \theta|}{5 \sec \theta} (5 \sec \theta \tan \theta) \, d\theta$$

$$= 5 \int \tan^2 \theta \, d\theta \quad \text{tan} \theta \geq 0 \text{ since } 0 \leq \theta < \pi/2$$

$$= 5 \int (\sec^2 \theta - 1) \, d\theta = 5 \tan \theta - 5 \theta + C$$

To express the solution in terms of $x$, we will represent the substitution $x = 5 \sec \theta$ geometrically by the triangle in Figure 7.4.5, from which we obtain

$$\tan \theta = \frac{\sqrt{x^2 - 25}}{5}$$

From this and the fact that the substitution can be expressed as $\theta = \sec^{-1}(x/5)$, we obtain

$$\int \frac{\sqrt{x^2 - 25}}{x} \, dx = \sqrt{x^2 - 25} - 5 \sec^{-1}\left(\frac{x}{5}\right) + C$$

INTEGRALS INVOLVING $ax^2 + bx + c$

Integrals that involve a quadratic expression $ax^2 + bx + c$, where $a \neq 0$ and $b \neq 0$, can often be evaluated by first completing the square, then making an appropriate substitution. The following example illustrates this idea.

Example 6 Evaluate $\int \frac{x}{x^2 - 4x + 8} \, dx$.

Solution. Completing the square yields

$$x^2 - 4x + 8 = (x^2 - 4x + 4) + 8 - 4 = (x - 2)^2 + 4$$

Thus, the substitution $u = x - 2, \quad du = dx$

yields

$$\int \frac{x}{x^2 - 4x + 8} \, dx = \int \frac{x}{(x - 2)^2 + 4} \, dx = \int \frac{u + 2}{u^2 + 4} \, du$$

$$= \int \frac{u}{u^2 + 4} \, du + 2 \int \frac{du}{u^2 + 4}$$

$$= \frac{1}{2} \int \frac{2u}{u^2 + 4} \, du + 2 \int \frac{du}{u^2 + 4}$$

$$= \frac{1}{2} \ln(u^2 + 4) + 2 \left(\frac{1}{2}\right) \tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{2} \ln[(x - 2)^2 + 4] + \tan^{-1}\left(\frac{x - 2}{2}\right) + C$$
7.4 Trigonometric Substitutions

**QUICK CHECK EXERCISES 7.4**
(See page 514 for answers.)

1. For each expression, give a trigonometric substitution that will eliminate the radical.
   (a) \(\sqrt{a^2 - x^2} \)  
   (b) \(\sqrt{a^2 + x^2} \)  
   (c) \(\sqrt{x^2 - a^2} \)

2. If \(x = 2 \sec \theta \) and \(0 < \theta < \pi/2\), then
   (a) \(\sin \theta = \)  
   (b) \(\cos \theta = \)  
   (c) \(\tan \theta = \)

3. In each part, state the trigonometric substitution that you would try first to evaluate the integral. Do not evaluate the integral.
   (a) \(\int \sqrt{9 + x^2} \, dx \)  
   (b) \(\int \sqrt{9 - x^2} \, dx \)  
   (c) \(\int \sqrt{1 - 9x^2} \, dx \)

4. In each part, determine the substitution \(u\).
   (a) \(\int \frac{1}{x^2 - 2x + 10} \, dx = \int \frac{1}{u^2 + 3^2} \, du; \)  
      \(u = \)  
   (b) \(\int \sqrt{x^2 - 6x + 8} \, dx = \int \sqrt{u^2 - 4} \, du; \)  
      \(u = \)  
   (c) \(\int \sqrt{12 - 4x - x^2} \, dx = \int \sqrt{4^2 - u^2} \, du; \)  
      \(u = \)

**EXERCISE SET 7.4**

1–26 Evaluate the integral.

1. \(\int \sqrt{4 - x^2} \, dx \)
2. \(\int \sqrt{1 - 4x^2} \, dx \)
3. \(\int \frac{x^2}{\sqrt{16 - x^2}} \, dx \)
4. \(\int \frac{dx}{x^2 \sqrt{9 - x^2}} \)
5. \(\int \frac{dx}{(4 + x^2)^3} \)
6. \(\int \frac{x^2}{\sqrt{5 + x^2}} \, dx \)
7. \(\int \frac{\sqrt{x^2 - 9}}{x} \, dx \)
8. \(\int \frac{dx}{x^2 \sqrt{x^2 - 16}} \)
9. \(\int \frac{3x^3}{\sqrt{1 - x^2}} \, dx \)
10. \(\int x^3 \sqrt{5 - x^2} \, dx \)
11. \(\int \frac{dx}{x^2 \sqrt{9x^2 - 4}} \)
12. \(\int \sqrt{1 + t^2} \, dt \)
13. \(\int \frac{dx}{(1 - x^2)^{3/2}} \)
14. \(\int \frac{dx}{x^2 \sqrt{x^2 + 25}} \)
15. \(\int \frac{dx}{\sqrt{x^2 - 9}} \)
16. \(\int \frac{dx}{1 + 2x^2 + x^4} \)
17. \(\int \frac{dx}{(4x^2 - 9)^{3/2}} \)
18. \(\int \frac{3x^3}{\sqrt{x^2 - 25}} \, dx \)
19. \(\int e^x \sqrt{1 - e^{2x}} \, dx \)
20. \(\int \frac{\cos \theta}{\sin^2 \theta} \, d\theta \)
21. \(\int_1^5 5x^3 \sqrt{1 - x^2} \, dx \)
22. \(\int_0^{1/2} \frac{dx}{(1 - x^2)^2} \)
23. \(\int_0^2 \frac{dx}{\sqrt{x^2 + x^2}} \)
24. \(\int_0^2 \frac{2 \sqrt{x^2 - 4}}{x} \, dx \)
25. \(\int_1^3 \frac{dx}{x^4 + 3} \)
26. \(\int_1^3 \frac{x^3}{(3 + x^2)^{3/2}} \, dx \)

27–30 **True–False** Determine whether the statement is true or false. Explain your answer.

27. An integrand involving a radical of the form \(\sqrt{a^2 - x^2}\) suggests the substitution \(x = a \sin \theta\).
28. The trigonometric substitution \(x = a \sin \theta\) is made with the restriction \(0 \leq \theta \leq \pi\).
29. An integrand involving a radical of the form \(\sqrt{x^2 - a^2}\) suggests the substitution \(x = a \cos \theta\).
30. The area enclosed by the ellipse \(x^2 + 4y^2 = 1\) is \(\pi/2\).

**FOCUS ON CONCEPTS**

31. The integral \(\int \frac{x}{x^2 + 4} \, dx\)

   can be evaluated either by a trigonometric substitution or by the substitution \(u = x^2 + 4\). Do it both ways and show that the results are equivalent.

32. The integral \(\int \frac{x^2}{x^2 + 4} \, dx\)

   can be evaluated either by a trigonometric substitution or by algebraically rewriting the numerator of the integrand as \((x^2 + 4) - 4\). Do it both ways and show that the results are equivalent.

33. Find the arc length of the curve \(y = \ln x\) from \(x = 1\) to \(x = 2\).
34. Find the arc length of the curve \(y = x^2\) from \(x = 0\) to \(x = 1\).
35. Find the area of the surface generated when the curve in Exercise 34 is revolved about the x-axis.

36. Find the volume of the solid generated when the region enclosed by \( y = (1 - y^2)^{1/4}, y = 0, y = 1, \) and \( x = 0 \) is revolved about the y-axis.

37–48 Evaluate the integral.

37. \( \int \frac{dx}{x^2 - 4x + 5} \)
38. \( \int \frac{dx}{\sqrt{2x - x^2}} \)
39. \( \int \frac{dx}{\sqrt{3 + 2x - x^2}} \)
40. \( \int \frac{dx}{16x^2 + 16 + 5} \)
41. \( \int \frac{dx}{\sqrt{x^2 - 6x + 10}} \)
42. \( \int \frac{x}{x^2 + 1 + 2x + 2} dx \)
43. \( \int \frac{dx}{\sqrt{5 - 2x - x^2}} \)
44. \( \int \frac{dx}{\sqrt{2} + e^x + e^{-x}} \)
45. \( \int \frac{dx}{\sqrt{2x^2 + 4x + 7}} \)
46. \( \int \frac{dx}{\sqrt{4x^2 + 4x + 5}} \)
47. \( \int \frac{dx}{\sqrt{x^4 - x^2}} \)
48. \( \int \frac{dx}{\sqrt{x(4 - x)}} \)

49–50 There is a good chance that your CAS will not be able to evaluate these integrals as stated. If this is so, make a substitution that converts the integral into one that your CAS can evaluate.

49. \( \int \cos x \sin x \sqrt{1 - \sin^2 x} \, dx \)

50. \( \int (x \cos x + \sin x) \sqrt{1 + x^2 \sin^2 x} \, dx \)

51. (a) Use the hyperbolic substitution \( x = 3 \sinh u, \) the identity \( \cosh^2 u - \sinh^2 u = 1, \) and Theorem 6.9.4 to evaluate

\[ \int \frac{dx}{\sqrt{x^2 + 9}} \]

(b) Evaluate the integral in part (a) using a trigonometric substitution and show that the result agrees with that obtained in part (a).

52. Use the hyperbolic substitution \( x = \cosh u, \) the identity \( \cosh^2 u - 1, \) and the results referenced in Exercise 51 to evaluate

\[ \int \sqrt{x^2 - 1} \, dx, \quad x \geq 1 \]

53. Writing The trigonometric substitution \( x = a \sin \theta, -\pi/2 \leq \theta \leq \pi/2, \) is suggested for an integral whose integrand involves \( \sqrt{a^2 - x^2}. \) Discuss the implications of restricting \( \theta \) to \( \pi/2 \leq \theta \leq 3\pi/2, \) and explain why the restriction \( -\pi/2 \leq \theta \leq \pi/2 \) should be preferred.

54. Writing The trigonometric substitution \( x = a \cos \theta \) could also be used for an integral whose integrand involves \( \sqrt{a^2 - x^2}. \) Determine an appropriate restriction for \( \theta \) with the substitution \( x = a \cos \theta, \) and discuss how to apply this substitution in appropriate integrals. Illustrate your discussion by evaluating the integral in Example 1 using a substitution of this type.

**QUICK CHECK ANSWERS 7.4**

1. (a) \( x = a \sin \theta \) (b) \( x = a \tan \theta \) (c) \( x = a \sec \theta \)
2. (a) \( \frac{\sqrt{x^2 - 4}}{x} \) (b) \( \frac{2}{x} \) (c) \( \frac{\sqrt{x^2 - 4}}{2} \)
3. (a) \( x = 3 \tan \theta \) (b) \( x = 3 \sin \theta \)
4. (a) \( x - 1 \) (b) \( x - 3 \) (c) \( x + 2 \)

**7.5 INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS**

Recall that a rational function is a ratio of two polynomials. In this section we will give a general method for integrating rational functions that is based on the idea of decomposing a rational function into a sum of simple rational functions that can be integrated by the methods studied in earlier sections.

**PARTIAL FRACTIONS**

In algebra, one learns to combine two or more fractions into a single fraction by finding a common denominator. For example,

\[
\frac{2}{x - 4} + \frac{3}{x + 1} = \frac{2(x + 1) + 3(x - 4)}{(x - 4)(x + 1)} = \frac{5x - 10}{x^2 - 3x - 4}
\]  (1)
7.5 Integrating Rational Functions by Partial Fractions

However, for purposes of integration, the left side of (1) is preferable to the right side since each of the terms is easy to integrate:

\[
\int \frac{5x - 10}{x^2 - 3x - 4} \, dx = \int \frac{2}{x - 4} \, dx + \int \frac{3}{x + 1} \, dx = 2 \ln |x - 4| + 3 \ln |x + 1| + C
\]

Thus, it is desirable to have some method that will enable us to obtain the left side of (1), starting with the right side. To illustrate how this can be done, we begin by noting that on the left side the numerators are constants and the denominators are the factors of the denominator on the right side. Thus, to find the left side of (1), starting from the right side, we could factor the denominator of the right side and look for constants \(A\) and \(B\) such that

\[
\frac{5x - 10}{(x - 4)(x + 1)} = \frac{A}{x - 4} + \frac{B}{x + 1}
\]

One way to find the constants \(A\) and \(B\) is to multiply (2) through by \((x - 4)(x + 1)\) to clear fractions. This yields

\[
5x - 10 = A(x + 1) + B(x - 4)
\]

This relationship holds for all \(x\), so it holds in particular if \(x = 4\) or \(x = -1\). Substituting \(x = 4\) in (3) makes the second term on the right drop out and yields the equation \(10 = 5A\) or \(A = 2\); and substituting \(x = -1\) in (3) makes the first term on the right drop out and yields the equation \(-15 = -5B\) or \(B = 3\). Substituting these values in (2) we obtain

\[
\frac{5x - 10}{(x - 4)(x + 1)} = \frac{2}{x - 4} + \frac{3}{x + 1}
\]

which agrees with (1).

A second method for finding the constants \(A\) and \(B\) is to multiply out the right side of (3) and collect like powers of \(x\) to obtain

\[
5x - 10 = (A + B)x + (A - 4B)
\]

Since the polynomials on the two sides are identical, their corresponding coefficients must be the same. Equating the corresponding coefficients on the two sides yields the following system of equations in the unknowns \(A\) and \(B\):

\[
\begin{align*}
A + B &= 5 \\
A - 4B &= -10
\end{align*}
\]

Solving this system yields \(A = 2\) and \(B = 3\) as before (verify).

The terms on the right side of (4) are called partial fractions of the expression on the left side because they each constitute part of that expression. To find those partial fractions we first had to make a guess about their form, and then we had to find the unknown constants. Our next objective is to extend this idea to general rational functions. For this purpose, suppose that \(P(x)/Q(x)\) is a proper rational function, by which we mean that the degree of the numerator is less than the degree of the denominator. There is a theorem in advanced algebra which states that every proper rational function can be expressed as a sum

\[
\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)
\]

where \(F_1(x), F_2(x), \ldots, F_n(x)\) are rational functions of the form

\[
\frac{A}{(ax + b)^k} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^k}
\]

in which the denominators are factors of \(Q(x)\). The sum is called the partial fraction decomposition of \(P(x)/Q(x)\), and the terms are called partial fractions. As in our opening example, there are two parts to finding a partial fraction decomposition: determining the exact form of the decomposition and finding the unknown constants.
FINDING THE FORM OF A PARTIAL FRACTION DECOMPOSITION

The first step in finding the form of the partial fraction decomposition of a proper rational function \( P(x)/Q(x) \) is to factor \( Q(x) \) completely into linear and irreducible quadratic factors, and then collect all repeated factors so that \( Q(x) \) is expressed as a product of distinct factors of the form

\[
(ax + b)^m \quad \text{and} \quad (ax^2 + bx + c)^m
\]

From these factors we can determine the form of the partial fraction decomposition using two rules that we will now discuss.

LINEAR FACTORS

If all of the factors of \( Q(x) \) are linear, then the partial fraction decomposition of \( P(x)/Q(x) \) can be determined by using the following rule:

**LINEAR FACTOR RULE**  
For each factor of the form \((ax + b)^m\), the partial fraction decomposition contains the following sum of \(m\) partial fractions:

\[
\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_m}{(ax + b)^m}
\]

where \(A_1, A_2, \ldots, A_m\) are constants to be determined. In the case where \(m = 1\), only the first term in the sum appears.

Example 1  
Evaluate \( \int \frac{dx}{x^2 + x - 2} \).

Solution.  
The integrand is a proper rational function that can be written as

\[
\frac{1}{x^2 + x - 2} = \frac{1}{(x - 1)(x + 2)}
\]

The factors \(x - 1\) and \(x + 2\) are both linear and appear to the first power, so each contributes one term to the partial fraction decomposition by the linear factor rule. Thus, the decomposition has the form

\[
\frac{1}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}
\]

where \(A\) and \(B\) are constants to be determined. Multiplying this expression through by \((x - 1)(x + 2)\) yields

\[
1 = A(x + 2) + B(x - 1)
\]

As discussed earlier, there are two methods for finding \(A\) and \(B\): we can substitute values of \(x\) that are chosen to make terms on the right drop out, or we can multiply out on the right and equate corresponding coefficients on the two sides to obtain a system of equations that can be solved for \(A\) and \(B\). We will use the first approach.

Setting \(x = 1\) makes the second term in (6) drop out and yields \(1 = 3A\) or \(A = \frac{1}{3}\); and setting \(x = -2\) makes the first term in (6) drop out and yields \(1 = -3B\) or \(B = -\frac{1}{3}\). Substituting these values in (5) yields the partial fraction decomposition

\[
\frac{1}{(x - 1)(x + 2)} = \frac{\frac{1}{3}}{x - 1} + \frac{-\frac{1}{3}}{x + 2}
\]
7.5 Integrating Rational Functions by Partial Fractions

The integration can now be completed as follows:

\[
\int \frac{dx}{(x - 1)(x + 2)} = \frac{1}{3} \int \frac{dx}{x - 1} - \frac{1}{3} \int \frac{dx}{x + 2} \\
= \frac{1}{3} \ln |x - 1| - \frac{1}{3} \ln |x + 2| + C = \frac{1}{3} \ln \left| \frac{x - 1}{x + 2} \right| + C \quad \blacktriangle
\]

If the factors of \( Q(x) \) are linear and none are repeated, as in the last example, then the recommended method for finding the constants in the partial fraction decomposition is to substitute appropriate values of \( x \) to make terms drop out. However, if some of the linear factors are repeated, then it will not be possible to find all of the constants in this way. In this case the recommended procedure is to find as many constants as possible by substitution and then find the rest by equating coefficients. This is illustrated in the next example.

Example 2 Evaluate \( \int \frac{2x + 4}{x^3 - 2x^2} \, dx \).

\textbf{Solution.} The integrand can be rewritten as

\[
\frac{2x + 4}{x^3 - 2x^2} = \frac{2x + 4}{x^2(x - 2)}
\]

Although \( x^2 \) is a quadratic factor, it is \textit{not} irreducible since \( x^2 = xx \). Thus, by the linear factor rule, \( x^2 \) introduces two terms (since \( m = 2 \)) of the form

\[
\frac{A}{x} + \frac{B}{x^2}
\]

and the factor \( x - 2 \) introduces one term (since \( m = 1 \)) of the form

\[
\frac{C}{x - 2}
\]

so the partial fraction decomposition is

\[
\frac{2x + 4}{x^3(x - 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 2} \quad (7)
\]

Multiplying by \( x^2(x - 2) \) yields

\[
2x + 4 = Ax(x - 2) + B(x - 2) + Cx^2 \quad (8)
\]

which, after multiplying out and collecting like powers of \( x \), becomes

\[
2x + 4 = (A + C)x^2 + (-2A + B)x - 2B \quad (9)
\]

Setting \( x = 0 \) in (8) makes the first and third terms drop out and yields \( B = -2 \), and setting \( x = 2 \) in (8) makes the first and second terms drop out and yields \( C = 2 \) (verify). However, there is no substitution in (8) that produces \( A \) directly, so we look to Equation (9) to find this value. This can be done by equating the coefficients of \( x^2 \) on the two sides to obtain

\[
A + C = 0 \quad \text{or} \quad A = -C = -2
\]

Substituting the values \( A = -2, \ B = -2, \) and \( C = 2 \) in (7) yields the partial fraction decomposition

\[
\frac{2x + 4}{x^3(x - 2)} = -\frac{2}{x} + -\frac{2}{x^2} + \frac{2}{x - 2}
\]

Thus,

\[
\int \frac{2x + 4}{x^3(x - 2)} \, dx = -2 \int \frac{dx}{x} - 2 \int \frac{dx}{x^2} + 2 \int \frac{dx}{x - 2} \\
= -2 \ln |x| + \frac{2}{x} + 2 \ln |x - 2| + C = 2 \ln \left| \frac{x - 2}{x} \right| + \frac{2}{x} + C \quad \blacktriangle
\]
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QUADRATIC FACTORS

If some of the factors of $Q(x)$ are irreducible quadratics, then the contribution of those factors to the partial fraction decomposition of $P(x)/Q(x)$ can be determined from the following rule:

**Quadratic Factor Rule**

For each factor of the form $(ax^2 + bx + c)^m$, the partial fraction decomposition contains the following sum of $m$ partial fractions:

$$
\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}
$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

Example 3

Evaluate $\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} \, dx$.

**Solution.** The denominator in the integrand can be factored by grouping:

$$3x^3 - x^2 + 3x - 1 = x^2(3x - 1) + (3x - 1) = (3x - 1)(x^2 + 1)$$

By the linear factor rule, the factor $3x - 1$ introduces one term, namely,

$$\frac{A}{3x - 1}$$

and by the quadratic factor rule, the factor $x^2 + 1$ introduces one term, namely,

$$\frac{Bx + C}{x^2 + 1}$$

Thus, the partial fraction decomposition is

$$\frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} = \frac{A}{3x - 1} + \frac{Bx + C}{x^2 + 1} \quad (10)$$

Multiplying by $(3x - 1)(x^2 + 1)$ yields

$$x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(3x - 1) \quad (11)$$

We could find $A$ by substituting $x = \frac{1}{3}$ to make the last term drop out, and then find the rest of the constants by equating corresponding coefficients. However, in this case it is just as easy to find all of the constants by equating coefficients and solving the resulting system. For this purpose we multiply out the right side of (11) and collect like terms:

$$x^2 + x - 2 = (A + 3B)x^2 + (-B + 3C)x + (A - C)$$

Equating corresponding coefficients gives

$$
\begin{align*}
A + 3B &= 1 \\
-B + 3C &= 1 \\
A - C &= -2
\end{align*}
$$

To solve this system, subtract the third equation from the first to eliminate $A$. Then use the resulting equation together with the second equation to solve for $B$ and $C$. Finally, determine $A$ from the first or third equation. This yields (verify)

$$A = -\frac{7}{5}, \quad B = \frac{4}{5}, \quad C = \frac{3}{5}$$
Thus, (10) becomes
\[
\frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} = \frac{-7}{3x - 1} + \frac{\frac{4}{5}x + \frac{3}{5}}{x^2 + 1}
\]
and
\[
\int \frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} \, dx = -\frac{7}{5} \int \frac{dx}{3x - 1} + \frac{4}{5} \int \frac{x}{x^2 + 1} \, dx + \frac{3}{5} \int \frac{dx}{x^2 + 1}
\]
\[
= -\frac{7}{15} \ln |3x - 1| + \frac{2}{5} \ln(x^2 + 1) + \frac{3}{5} \tan^{-1} x + C \tag{12}
\]

**Example 4** Evaluate \( \int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x + 2)(x^2 + 3)^2} \, dx \).

**Solution.** Observe that the integrand is a proper rational function since the numerator has degree 4 and the denominator has degree 5. Thus, the method of partial fractions is applicable. By the linear factor rule, the factor \( x + 2 \) introduces the single term
\[
\frac{A}{x + 2}
\]
and by the quadratic factor rule, the factor \((x^2 + 3)^2\) introduces two terms (since \( m = 2 \)): \[
\frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2 + 3)^2}
\]
Thus, the partial fraction decomposition of the integrand is
\[
\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x + 2)(x^2 + 3)^2} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2 + 3)^2} \tag{12}
\]
Multiplying by \((x + 2)(x^2 + 3)^2\) yields
\[
3x^4 + 4x^3 + 16x^2 + 20x + 9 = A(x^2 + 3)^2 + (Bx + C)(x^2 + 3)(x + 2) + (Dx + E)(x + 2) \tag{13}
\]
which, after multiplying out and collecting like powers of \( x \), becomes
\[
3x^4 + 4x^3 + 16x^2 + 20x + 9 = (A + B)x^4 + (2B + C)x^3 + (6A + 3B + 2C + D)x^2 + (6B + 3C + 2D + E)x + (9A + 6C + 2E) \tag{14}
\]
Equating corresponding coefficients in (14) yields the following system of five linear equations in five unknowns:
\[
\begin{align*}
A + B &= 3 \\
2B + C &= 4 \\
6A + 3B + 2C + D &= 16 \\
6B + 3C + 2D + E &= 20 \\
9A + 6C + 2E &= 9
\end{align*} \tag{15}
\]
Efficient methods for solving systems of linear equations such as this are studied in a branch of mathematics called **linear algebra**; those methods are outside the scope of this text. However, as a practical matter most linear systems of any size are solved by computer, and most computer algebra systems have commands that in many cases can solve linear systems exactly. In this particular case we can simplify the work by first substituting \( x = -2 \)
in (13), which yields $A = 1$. Substituting this known value of $A$ in (15) yields the simpler system

$$
\begin{aligned}
B &= 2 \\
2B + C &= 4 \\
3B + 2C + D &= 10 \\
6B + 3C + 2D + E &= 20 \\
6C + 2E &= 0
\end{aligned}
$$

(16)

This system can be solved by starting at the top and working down, first substituting $B = 2$ in the second equation to get $C = 0$, then substituting the known values of $B$ and $C$ in the third equation to get $D = 4$, and so forth. This yields

$$
A = 1, \quad B = 2, \quad C = 0, \quad D = 4, \quad E = 0
$$

Thus, (12) becomes

$$
\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x + 2)(x^2 + 3)^2} = \frac{1}{x + 2} + \frac{2x}{x^2 + 3} + \frac{4x}{(x^2 + 3)^2}
$$

and so

$$
\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x + 2)(x^2 + 3)^2} \, dx = \int \frac{dx}{x + 2} + \int \frac{2x}{x^2 + 3} \, dx + 4 \int \frac{x}{(x^2 + 3)^2} \, dx
$$

$$
= \ln |x + 2| + \ln(x^2 + 3) - \frac{2}{x^2 + 3} + C
$$

\section{INTEGRATING IMPROPER RATIONAL FUNCTIONS}

Although the method of partial fractions only applies to proper rational functions, an improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function, which can then be decomposed into partial fractions. This idea is illustrated in the following example.

\begin{example}
Evaluate $\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} \, dx$.
\end{example}

\textbf{Solution.} The integrand is an improper rational function since the numerator has degree 4 and the denominator has degree 2. Thus, we first perform the long division

$$
\begin{align*}
x^2 + x - 2 & \bigg| \begin{array}{c}
3x^4 + 3x^3 - 5x^2 + x - 1 \\
3x^4 + 3x^3 - 6x^2 \\
x^2 + x - 1 \\
x^2 + x - 2
\end{array}
\end{align*}
$$

It follows that the integrand can be expressed as

$$
\frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} = (3x^2 + 1) + \frac{1}{x^2 + x - 2}
$$

and hence

$$
\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} \, dx = \int (3x^2 + 1) \, dx + \int \frac{dx}{x^2 + x - 2}
$$
7.5 Integrating Rational Functions by Partial Fractions

The second integral on the right now involves a proper rational function and can thus be evaluated by a partial fraction decomposition. Using the result of Example 1 we obtain
\[ \int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} \, dx = x^3 + x + \frac{1}{3} \ln \left| \frac{x - 1}{x + 2} \right| + C \]

**CONCLUDING REMARKS**

There are some cases in which the method of partial fractions is inappropriate. For example, it would be inefficient to use partial fractions to perform the integration
\[ \int \frac{3x^2 + 2}{x^3 + 2x - 8} \, dx = \ln |x^3 + 2x - 8| + C \]

since the substitution \( u = x^3 + 2x - 8 \) is more direct. Similarly, the integration
\[ \int \frac{2x - 1}{x^2 + 1} \, dx = \int \frac{2x}{x^2 + 1} \, dx - \int \frac{dx}{x^2 + 1} = \ln(x^2 + 1) - \tan^{-1} x + C \]

requires only a little algebra since the integrand is already in partial fraction form.

**QUICK CHECK EXERCISES 7.5** *(See page 523 for answers.)*

1. A partial fraction is a rational function of the form ________ or of the form ________.

2. (a) What is a proper rational function?
   (b) What condition must the degree of the numerator and the degree of the denominator of a rational function satisfy for the method of partial fractions to be applicable directly?
   (c) If the condition in part (b) is not satisfied, what must you do if you want to use partial fractions?

3. Suppose that the function \( f(x) = P(x)/Q(x) \) is a proper rational function.
   (a) For each factor of \( Q(x) \) of the form \( (ax + b)^m \), the partial fraction decomposition of \( f \) contains the following sum of \( m \) partial fractions: ________

4. Complete the partial fraction decomposition.
   (a) \[ \frac{-3}{(x + 1)(2x - 1)} = \frac{A}{x + 1} - \frac{2}{2x - 1} \]
   (b) \[ \frac{2x^2 - 3x}{(x^2 + 1)(3x + 2)} = \frac{B}{3x + 2} - \frac{1}{x^2 + 1} \]

5. Evaluate the integral.
   (a) \[ \int \frac{3}{(x + 1)(1 - 2x)} \, dx \]
   (b) \[ \int \frac{2x^2 - 3x}{(x^2 + 1)(3x + 2)} \, dx \]

**EXERCISE SET 7.5**

1–8 Write out the form of the partial fraction decomposition. (Do not find the numerical values of the coefficients.)

1. \[ \frac{3x - 1}{(x - 3)(x + 4)} \]
2. \[ \frac{5}{x(x^2 - 4)} \]
3. \[ \frac{2x - 3}{x^3 - x^2} \]
4. \[ \frac{x^2}{(x + 2)^2} \]
5. \[ \frac{1 - x^2}{x^3(x^2 + 2)} \]
6. \[ \frac{3x}{(x - 1)(x^2 + 6)} \]
7. \[ \frac{4x^3 - x}{(x^2 + 5)^2} \]
8. \[ \frac{1 - 3x^4}{(x - 2)(x^2 + 1)^2} \]

9–34 Evaluate the integral.

9. \[ \int \frac{dx}{x^2 - 3x - 4} \]
10. \[ \int \frac{dx}{x^2 - 6x - 7} \]
11. \[ \int \frac{11x + 17}{2x^2 + 7x - 4} \, dx \]
12. \[ \int \frac{5x - 5}{3x^2 - 8x - 3} \, dx \]
13. \[ \int \frac{2x^2 - 9x - 9}{x^3 - 9x} \, dx \]
14. \[ \int \frac{dx}{x(x^2 - 1)} \]
15. \[ \int \frac{x^2 - 8}{x + 3} \, dx \]
16. \[ \int \frac{x^2 + 1}{x - 1} \, dx \]
17. \[ \int \frac{3x^2 - 10}{x^3 - 4x + 4} \, dx \]
18. \[ \int \frac{dx}{x^2 - 3x + 2} \]
19. \[ \int \frac{2x - 3}{x^2 - 3x - 10} \, dx \]
20. \[ \int \frac{3x + 1}{3x^2 + 2x - 1} \, dx \]
21. \[ \int \frac{x^3 + x^2 + 2}{x^3 - x} \, dx \]
22. \[ \int \frac{x^3 - 4x^2 + 1}{x^3 - 4x} \, dx \]
23. \[ \int \frac{2x^2 + 3}{x(x - 1)^2} \, dx \]
24. \[ \int \frac{3x^2 - x + 1}{x^3 - x^2} \, dx \]
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25. \( \int \frac{2x^2 - 10x + 4}{x + 1}(x - 3)^2 \, dx \)
26. \( \int \frac{2x^2 - 2x - 1}{x^3 - x^2} \, dx \)
27. \( \int \frac{x^7}{(x + 1)^3} \, dx \)
28. \( \int \frac{2x^2 + 3x + 3}{(x + 1)^3} \, dx \)
29. \( \int \frac{2x^3 - 1}{(4x - 1)(x^2 + 1)} \, dx \)
30. \( \int \frac{dx}{x^3 + 2x} \)
31. \( \int \frac{x^3 + 3x^2 + x + 9}{(x + 1)(x^2 + 3)} \, dx \)
32. \( \int \frac{x^3 + x^2 + x + 2}{(x^2 + 1)(x + 2)} \, dx \)
33. \( \int \frac{x^3 - 2x^2 + 2x - 2}{x^2 + 1} \, dx \)
34. \( \int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10} \, dx \)

35–38 True-False. Determine whether the statement is true or false. Explain your answer. ■

35. The technique of partial fractions is used for integrals whose integrands are ratios of polynomials.
36. The integrand in \( \int \frac{3x^4 + 5}{(x^2 + 1)^2} \, dx \) is a proper rational function.
37. The partial fraction decomposition of \( \frac{2x + 3}{x^2} \) is \( \frac{2}{x} + \frac{3}{x^2} \).
38. If \( f(x) = \frac{P(x)}{(x + 5)^3} \) is a proper rational function, then the partial fraction decomposition of \( f(x) \) has terms with constant numerators and denominators \( (x + 5) \), \( (x + 5)^2 \), and \( (x + 5)^3 \).

39–42 Evaluate the integral by making a substitution that converts the integrand to a rational function. ■

39. \( \int \frac{\cos \theta}{\sin^2 \theta + 4 \sin \theta - 5} \, d\theta \)
40. \( \int \frac{e^{t^2}}{5 - 4t} \, dt \)
41. \( \int \frac{e^{x^3}}{x} \, dx \)
42. \( \int \frac{5 + 2 \ln x}{x(x + 1 \ln x)^2} \, dx \)

43. Find the volume of the solid generated when the region enclosed by \( y = x^2/(9 - x^2) \), \( y = 0 \), \( x = 0 \), and \( x = 2 \) is revolved about the x-axis.
44. Find the area of the region under the curve \( y = 1/(1 + e^t) \), over the interval \([-\ln 5, \ln 5] \). [Hint: Make a substitution that converts the integrand to a rational function.]

45–46 Use a CAS to evaluate the integral in two ways: (i) integrate directly; (ii) use the CAS to find the partial fraction decomposition and integrate the decomposition. Integrate by hand to check the results. ■

45. \( \int \frac{x^2 + 1}{(x^2 + 2x + 3)^2} \, dx \)
46. \( \int \frac{x^5 + x^4 + 4x^3 + 4x^2 + 4}{(x^2 + 2)^3} \, dx \)

47–48 Integrate by hand and check your answers using a CAS.

47. \( \int \frac{dx}{x^4 - 3x^3 - 7x^2 + 27x - 18} \)
48. \( \int \sqrt{16x^3 - 4x^2 + 4x - 1} \)

FOCUS ON CONCEPTS

49. Show that \( \int_0^1 \frac{x}{x^4 + 1} \, dx = \frac{\pi}{8} \)
50. Use partial fractions to derive the integration formula \( \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C \)
51. Suppose that \( ax^2 + bx + c \) is a quadratic polynomial and that the integration \( \int \frac{1}{ax^2 + bx + c} \, dx \) produces a function with no inverse tangent terms. What does this tell you about the roots of the polynomial?
52. Suppose that \( ax^2 + bx + c \) is a quadratic polynomial and that the integration \( \int \frac{1}{ax^2 + bx + c} \, dx \) produces a function with neither logarithmic nor inverse tangent terms. What does this tell you about the roots of the polynomial?
53. Does there exist a quadratic polynomial \( ax^2 + bx + c \) such that the integration \( \int \frac{x}{ax^2 + bx + c} \, dx \) produces a function with no logarithmic terms? If so, give an example; if not, explain why no such polynomial can exist.

54. Writing Suppose that \( P(x) \) is a cubic polynomial. State the general form of the partial fraction decomposition for \( f(x) = \frac{P(x)}{(x + 5)^3} \) and state the implications of this decomposition for evaluating the integral \( \int f(x) \, dx \).

55. Writing Consider the functions \( f(x) = \frac{1}{x^2 - 4} \) and \( g(x) = \frac{x}{x^2 - 4} \).
Each of the integrals \( \int f(x) \, dx \) and \( \int g(x) \, dx \) can be evaluated using partial fractions and using at least one other integration technique. Demonstrate two different techniques for evaluating each of these integrals, and then discuss the considerations that would determine which technique you would use.
Quick Check Answers 7.5

1. \( \frac{A}{(ax+b)^2}; \frac{Ax+B}{(ax^2+bx+c)^2} \)

2. (a) A proper rational function is a rational function in which the degree of the numerator is less than the degree of the denominator.
   (b) The degree of the numerator must be less than the degree of the denominator.
   (c) Divide the denominator into the numerator, which results in the sum of a polynomial and a proper rational function.

3. (a) \( \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_m}{(ax+b)^m} \)
   (b) \( \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_mx+B_m}{(ax^2+bx+c)^m} \)

4. (a) \( A = 1 \) (b) \( B = 2 \)

5. (a) \( \int \frac{3}{(x+1)(1-2x)} \, dx = \ln \left| \frac{x+1}{1-2x} \right| + C \)
   (b) \( \int \frac{2x^2-3x}{(x^2+1)(3x+2)} \, dx = \frac{2}{3} \ln |3x+2| - \tan^{-1}x + C \)

7.6 Using Computer Algebra Systems and Tables of Integrals

In this section we will discuss how to integrate using tables, and we will see some special substitutions to try when an integral doesn’t match any of the forms in an integral table. In particular, we will discuss a method for integrating rational functions of \( \sin x \) and \( \cos x \). We will also address some of the issues that relate to using computer algebra systems for integration. Readers who are not using computer algebra systems can skip that material.

Intrinsic Tables

Tables of integrals are useful for eliminating tedious hand computation. The endpapers of this text contain a relatively brief table of integrals that we will refer to as the Endpaper Integral Table; more comprehensive tables are published in standard reference books such as the CRC Standard Mathematical Tables and Formulae, CRC Press, Inc., 2002.

All integral tables have their own scheme for classifying integrals according to the form of the integrand. For example, the Endpaper Integral Table classifies the integrals into 15 categories; Basic Functions, Reciprocals of Basic Functions, Powers of Trigonometric Functions, Products of Trigonometric Functions, and so forth. The first step in working with tables is to read through the classifications so that you understand the classification scheme and know where to look in the table for integrals of different types.

Perfect Matches

If you are lucky, the integral you are attempting to evaluate will match up perfectly with one of the forms in the table. However, when looking for matches you may have to make an adjustment for the variable of integration. For example, the integral

\[ \int x^2 \sin x \, dx \]

is a perfect match with Formula (46) in the Endpaper Integral Table, except for the letter used for the variable of integration. Thus, to apply Formula (46) to the given integral we need to change the variable of integration in the formula from \( u \) to \( x \). With that minor modification we obtain

\[ \int x^2 \sin x \, dx = 2x \sin x + (2-x^2) \cos x + C \]

Here are some more examples of perfect matches.
Example 1  Use the Endpaper Integral Table to evaluate

(a) \( \int \sin 7 x \cos 2 x \, dx \)  \hspace{1cm} (b) \( \int x^2 \sqrt{7 + 3x} \, dx \)

(c) \( \int \frac{\sqrt{2 - x^2}}{x} \, dx \)  \hspace{1cm} (d) \( \int (x^3 + 7x + 1) \sin \pi x \, dx \)

Solution (a). The integrand can be classified as a product of trigonometric functions. Thus, from Formula (40) with \( m = 7 \) and \( n = 2 \) we obtain

\[
\int \sin 7 x \cos 2 x \, dx = -\frac{\cos 9x}{18} - \frac{\cos 5x}{10} + C
\]

Solution (b). The integrand can be classified as a power of \( x \) multiplying \( \sqrt{a+bx} \). Thus, from Formula (103) with \( a = 7 \) and \( b = 3 \) we obtain

\[
\int x^2 \sqrt{7 + 3x} \, dx = \frac{2}{2885} (135x^2 - 252x + 392)(7 + 3x)^{3/2} + C
\]

Solution (c). The integrand can be classified as a power of \( x \) dividing \( \sqrt{a^2 - x^2} \). Thus, from Formula (79) with \( a = \sqrt{2} \) we obtain

\[
\int \frac{\sqrt{2 - x^2}}{x} \, dx = \sqrt{2 - x^2} - \sqrt{2} \ln \left| \frac{\sqrt{2} + \sqrt{2 - x^2}}{x} \right| + C
\]

Solution (d). The integrand can be classified as a polynomial multiplying a trigonometric function. Thus, we apply Formula (58) with \( p(x) = x^3 + 7x + 1 \) and \( a = \pi \). The successive nonzero derivatives of \( p(x) \) are

\[
p'(x) = 3x^2 + 7, \quad p''(x) = 6x, \quad p'''(x) = 6
\]

and so

\[
\int (x^3 + 7x + 1) \sin \pi x \, dx = -\frac{x^3}{\pi} \cos \pi x + \frac{3x^2}{\pi^2} \sin \pi x + \frac{6x}{\pi^3} \cos \pi x - \frac{6}{\pi^4} \sin \pi x + C
\]

Matches Requiring Substitutions

Sometimes an integral that does not match any table entry can be made to match by making an appropriate substitution.

Example 2  Use the Endpaper Integral Table to evaluate

(a) \( \int e^{x^3} \sin^{-1}(e^{x^3}) \, dx \)  \hspace{1cm} (b) \( \int x \sqrt{x^2 - 4x + 5} \, dx \)

Solution (a). The integrand does not even come close to matching any of the forms in the table. However, a little thought suggests the substitution

\[
u = e^{x^3}, \quad du = 3xe^{x^3} \, dx
\]
from which we obtain
\[ \int e^{\pi x} \sin^{-1}(e^{\pi x}) \, dx = \frac{1}{\pi} \int \sin^{-1} u \, du \]
The integrand is now a basic function, and Formula (7) yields
\[ \int e^{\pi x} \sin^{-1}(e^{\pi x}) \, dx = \frac{1}{\pi} \left[ u \sin^{-1} u + \sqrt{1 - u^2} \right] + C \]
\[ = \frac{1}{\pi} \left[ e^{\pi x} \sin^{-1}(e^{\pi x}) + \sqrt{1 - e^{2\pi x}} \right] + C \]

**Solution (b).** Again, the integrand does not closely match any of the forms in the table. However, a little thought suggests that it may be possible to bring the integrand closer to the form \( x \sqrt{x^2 + a^2} \) by completing the square to eliminate the term involving \( x \) inside the radical. Doing this yields
\[ \int x \sqrt{x^2 - 4x + 5} \, dx = \int \sqrt{(x^2 - 4x + 4) + 1} \, dx = \int \sqrt{(x - 2)^2 + 1} \, dx \quad (1) \]
At this point we are closer to the form \( x \sqrt{x^2 + a^2} \), but we are not quite there because of the \((x - 2)^2\) rather than \(x^2\) inside the radical. However, we can resolve that problem with the substitution
\[ u = x - 2, \quad du = dx \]
With this substitution we have \( x = u + 2 \), so (1) can be expressed in terms of \( u \) as
\[ \int (u + 2) \sqrt{u^2 + 1} \, du = \int u \sqrt{u^2 + 1} \, du + 2 \int \sqrt{u^2 + 1} \, du \]
The first integral on the right is now a perfect match with Formula (84) with \( a = 1 \), and the second is a perfect match with Formula (72) with \( a = 1 \). Thus, applying these formulas we obtain
\[ \int x \sqrt{x^2 - 4x + 5} \, dx = \frac{1}{2} (u^2 + 1)^{3/2} + 2 \left[ \frac{1}{2} u \sqrt{u^2 + 1} + \frac{1}{2} \ln(u + \sqrt{u^2 + 1}) \right] + C \]
If we now replace \( u \) by \( x - 2 \) (in which case \( u^2 + 1 = x^2 - 4x + 5 \)), we obtain
\[ \int x \sqrt{x^2 - 4x + 5} \, dx = \frac{1}{2} (x^2 - 4x + 5)^{3/2} + \frac{1}{2} (x - 2) \sqrt{x^2 - 4x + 5} + \ln \left( x - 2 + \sqrt{x^2 - 4x + 5} \right) + C \]
Although correct, this form of the answer has an unnecessary mixture of radicals and fractional exponents. If desired, we can “clean up” the answer by writing
\[ (x^2 - 4x + 5)^{3/2} = (x^2 - 4x + 5) \sqrt{x^2 - 4x + 5} \]
from which it follows that (verify)
\[ \int x \sqrt{x^2 - 4x + 5} \, dx = \frac{1}{4} (x^2 - x - 1) \sqrt{x^2 - 4x + 5} + \ln \left( x - 2 + \sqrt{x^2 - 4x + 5} \right) + C \]

**MATCHES REQUIRING REDUCTION FORMULAS**
In cases where the entry in an integral table is a reduction formula, that formula will have to be applied first to reduce the given integral to a form in which it can be evaluated.
Example 3  Use the Endpaper Integral Table to evaluate \( \int \frac{x^3}{\sqrt{1 + x}} \, dx \).

Solution.  The integrand can be classified as a power of \( x \) multiplying the reciprocal of \( \sqrt{a + bx} \). Thus, from Formula (107) with \( a = 1, b = 1, \) and \( n = 3 \), followed by Formula (106), we obtain
\[
\int \frac{x^3}{\sqrt{1 + x}} \, dx = \frac{2x^3\sqrt{1 + x}}{7} - \frac{6}{7} \int \frac{x^2}{\sqrt{1 + x}} \, dx
\]
\[
= \frac{2x^3\sqrt{1 + x}}{7} - \frac{6}{7} \left[ \frac{2}{15} (3x^2 - 4x + 8) \sqrt{1 + x} \right] + C
\]
\[
= \left( \frac{2x^3}{7} - \frac{12x^2}{35} + \frac{16x}{35} - \frac{32}{35} \right) \sqrt{1 + x} + C \]

SPECIAL SUBSTITUTIONS

The Endpaper Integral Table has numerous entries involving an exponent of \( 3/2 \) or involving square roots (exponent \( 1/2 \)), but it has no entries with other fractional exponents. However, integrals involving fractional powers of \( x \) can often be simplified by making the substitution \( u = x^{1/n} \) in which \( n \) is the least common multiple of the denominators of the exponents. The resulting integral will then involve integer powers of \( u \).

Example 4  Evaluate

(a) \( \int \frac{\sqrt{x}}{1 + \sqrt{x}} \, dx \)  
(b) \( \int \sqrt{1 + e^x} \, dx \)

Solution (a).  The integrand contains \( x^{1/2} \) and \( x^{1/3} \), so we make the substitution \( u = x^{1/6} \), from which we obtain
\[
x = u^6, \quad dx = 6u^5 \, du
\]
Thus,
\[
\int \frac{\sqrt{x}}{1 + \sqrt{x}} \, dx = \int \frac{(u^6)^{1/2}}{1 + (u^6)^{1/3}} (6u^5) \, du = 6 \int \frac{u^8}{1 + u^2} \, du
\]
By long division
\[
u^8 + u^2 = u^6 - u^4 + u^2 - 1 + \frac{1}{1 + u^2}
\]
from which it follows that
\[
\int \frac{\sqrt{x}}{1 + \sqrt{x}} \, dx = 6 \int \left( u^6 - u^4 + u^2 - 1 + \frac{1}{1 + u^2} \right) \, du
\]
\[
= \frac{6}{7} u^7 - \frac{6}{5} u^5 + 2u^3 - 6u + 6 \tan^{-1} u + C
\]
\[
= \frac{6}{7} x^{7/6} - \frac{6}{5} x^{5/6} + 2x^{1/2} - 6x^{1/6} + 6 \tan^{-1} (x^{1/6}) + C
\]

Solution (b).  The integral does not match any of the forms in the Endpaper Integral Table. However, the table does include several integrals containing \( \sqrt{a + bu} \). This suggests the substitution \( u = e^x \), from which we obtain
\[
x = \ln u, \quad dx = \frac{1}{u} \, du
Thus, from Formula (110) with $a = 1$ and $b = 1$, followed by Formula (108), we obtain

$$
\int \sqrt{1 + e^x} \, dx = \int \frac{\sqrt{1 + u}}{u} \, du
$$

$$
= 2\sqrt{1 + u} + \int \frac{du}{\sqrt{1 + u}}
$$

$$
= 2\sqrt{1 + u} + \ln\left| \frac{\sqrt{1 + u} - 1}{\sqrt{1 + u} + 1} \right| + C
$$

$$
= 2\sqrt{1 + e^x} + \ln\left[ \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right] + C
$$

Functions that consist of finitely many sums, differences, quotients, and products of $\sin x$ and $\cos x$ are called rational functions of $\sin x$ and $\cos x$. Some examples are

- $\frac{\sin x + 3 \cos^3 x}{\cos x + 4 \sin x}$
- $\frac{\sin x}{1 + \cos x - \cos^2 x}$
- $\frac{3 \sin^3 x}{1 + 4 \sin x}$

The Endpaper Integral Table gives a few formulas for integrating rational functions of $\sin x$ and $\cos x$ under the heading Reciprocals of Basic Functions. For example, it follows from Formula (18) that

$$
\int \frac{1}{1 + \sin x} \, dx = \tan x - \sec x + C
$$

(2)

However, since the integrand is a rational function of $\sin x$, it may be desirable in a particular application to express the value of the integral in terms of $\sin x$ and $\cos x$ and rewrite (2) as

$$
\int \frac{1}{1 + \sin x} \, dx = \frac{\sin x - 1}{\cos x} + C
$$

Many rational functions of $\sin x$ and $\cos x$ can be evaluated by an ingenious method that was discovered by the mathematician Karl Weierstrass (see p. 102 for biography). The idea is to make the substitution

$$
u = \tan(x/2), \quad -\pi/2 < x/2 < \pi/2$$

from which it follows that

$$
x = 2\tan^{-1} u, \quad dx = \frac{2}{1 + u^2} \, du
$$

To implement this substitution we need to express $\sin x$ and $\cos x$ in terms of $u$. For this purpose we will use the identities

$$
\sin x = 2 \sin(x/2) \cos(x/2)
$$

$$
\cos x = \cos^2(x/2) - \sin^2(x/2)
$$

(3)

(4)

and the following relationships suggested by Figure 7.6.1:

$$
\sin(x/2) = \frac{u}{\sqrt{1 + u^2}} \quad \text{and} \quad \cos(x/2) = \frac{1}{\sqrt{1 + u^2}}
$$

Substituting these expressions in (3) and (4) yields

$$
\sin x = 2 \left( \frac{u}{\sqrt{1 + u^2}} \right) \left( \frac{1}{\sqrt{1 + u^2}} \right) = \frac{2u}{1 + u^2}
$$

$$
\cos x = \left( \frac{1}{\sqrt{1 + u^2}} \right)^2 - \left( \frac{u}{\sqrt{1 + u^2}} \right)^2 = \frac{1 - u^2}{1 + u^2}
$$
In summary, we have shown that the substitution \( u = \tan(x/2) \) can be implemented in a rational function of \( \sin x \) and \( \cos x \) by letting
\[
\sin x = \frac{2u}{1 + u^2}, \quad \cos x = \frac{1 - u^2}{1 + u^2}, \quad dx = \frac{2}{1 + u^2} \, du \tag{5}
\]

**Example 5** Evaluate \( \int \frac{dx}{1 - \sin x + \cos x} \).

**Solution.** The integrand is a rational function of \( \sin x \) and \( \cos x \) that does not match any of the formulas in the Endpaper Integral Table, so we make the substitution \( u = \tan(x/2) \).

The substitution \( u = \tan(x/2) \) will convert any rational function of \( \sin x \) and \( \cos x \) to an ordinary rational function of \( u \). However, the method can lead to cumbersome partial fraction decompositions, so it may be worthwhile to consider other methods as well when hand computations are being used.

\[
\int \frac{dx}{1 - \sin x + \cos x} = \int \frac{2du}{1 + u^2} = \int \frac{2du}{(1 + u^2) - 2u + (1 - u^2)} = \int \frac{du}{1 - u} = -\ln |1 - u| + C = -\ln |1 - \tan(x/2)| + C \tag{6}
\]

**INTEGRATING WITH COMPUTER ALGEBRA SYSTEMS**

Integration tables are rapidly giving way to computerized integration using computer algebra systems. However, as with many powerful tools, a knowledgeable operator is an important component of the system.

Sometimes computer algebra systems do not produce the most general form of the indefinite integral. For example, the integral formula
\[
\int \frac{dx}{x - 1} = \ln |x - 1| + C
\]

which can be obtained by inspection or by using the substitution \( u = x - 1 \), is valid for \( x > 1 \) or for \( x < 1 \). However, not all computer algebra systems produce this form of the answer. Some typical answers produced by various implementations of *Mathematica*, *Maple*, and the CAS on a handheld calculator are

\[
\ln(-1 + x), \quad \ln(x - 1), \quad \ln(|x - 1|)
\]

Observe that none of the systems include the constant of integration—the answer produced is a particular antiderivative and not the most general antiderivative (indefinite integral). Observe also that only one of these answers includes the absolute value signs; the antiderivatives produced by the other systems are valid only for \( x > 1 \). All systems, however, are able to calculate the definite integral
\[
\int_{0}^{1/2} \frac{dx}{x - 1} = -\ln 2
\]
correctly. Now let us examine how these systems handle the integral
\[
\int x \sqrt{x^2 - 4x + 5} \, dx = \frac{1}{4} (x^2 - x - 1) \sqrt{x^2 - 4x + 5} + \ln(x - 2 + \sqrt{x^2 - 4x + 5}) \tag{6}
\]
7.6 Using Computer Algebra Systems and Tables of Integrals

which we obtained in Example 2(b) (with the constant of integration included). Some CAS implementations produce this result in slightly different algebraic forms, but a version of Maple produces the result

\[ \int x\sqrt{x^2 - 4x + 5} \, dx = \frac{1}{4} (x^2 - 4x + 5)^{3/2} + \frac{1}{2} (2x - 4) \sqrt{x^2 - 4x + 5} + \sinh^{-1}(x - 2) \]

This can be rewritten as (6) by expressing the fractional exponent in radical form and expressing \( \sinh^{-1}(x - 2) \) in logarithmic form using Theorem 6.9.4 (verify). A version of Mathematica produces the result

\[ \int x\sqrt{x^2 - 4x + 5} \, dx = \frac{1}{2} (x^2 - x - 1) \sqrt{x^2 - 4x + 5} - \sinh^{-1}(2 - x) \]

which can be rewritten in form (6) by using Theorem 6.9.4 together with the identity \( \sinh^{-1}(-x) = -\sinh^{-1}x \) (verify).

Computer algebra systems can sometimes produce inconvenient or unnatural answers to integration problems. For example, various computer algebra systems produced the following results when asked to integrate \((x + 1)^2\):

\[ (x + 1)^2 = \frac{1}{8} x^8 + x^7 + \frac{7}{2} x^6 + 7x^5 + \frac{35}{4} x^4 + 7x^3 + \frac{7}{2} x^2 + x \] (7)

The first form is in keeping with the hand computation

\[ \int (x + 1)^7 \, dx = \frac{(x + 1)^8}{8} + C \]

that uses the substitution \( u = x + 1 \), whereas the second form is based on expanding \((x + 1)^7\) and integrating term by term.

In Example 2(a) of Section 7.3 we showed that

\[ \int \sin^4 x \cos^5 x \, dx = \frac{1}{8} \sin^5 x - \frac{1}{7} \sin^7 x + \frac{1}{2} \sin^9 x + C \]

However, a version of Mathematica integrates this as

\[ \frac{1}{128} \sin x - \frac{1}{192} \sin 3x - \frac{1}{320} \sin 5x + \frac{1}{1792} \sin 7x + \frac{1}{2804} \sin 9x \]

whereas other computer algebra systems essentially integrate it as

\[ -\frac{1}{9} \sin^3 x \cos^6 x - \frac{1}{3} \sin x \cos^6 x + \frac{1}{105} \cos^4 x \sin x + \frac{4}{315} \cos^2 x \sin x + \frac{8}{315} \sin x \]

Although these three results look quite different, they can be obtained from one another using appropriate trigonometric identities.

### COMPUTER ALGEBRA SYSTEMS HAVE LIMITATIONS

A computer algebra system combines a set of integration rules (such as substitution) with a library of functions that it can use to construct antiderivatives. Such libraries contain elementary functions, such as polynomials, rational functions, trigonometric functions, as well as various nonelementary functions that arise in engineering, physics, and other applied fields. Just as our Endpaper Integral Table has only 121 indefinite integrals, these libraries are not exhaustive of all possible integrands. If the system cannot manipulate the integrand to a form matching one in its library, the program will give some indication that it cannot evaluate the integral. For example, when asked to evaluate the integral

\[ \int (1 + \ln x)\sqrt{1 + (x \ln x)^2} \, dx \] (8)

all of the systems mentioned above respond by displaying some form of the unevaluated integral as an answer, indicating that they could not perform the integration.

Sometimes computer algebra systems respond by expressing an integral in terms of another integral. For example, if you try to integrate \( e^{x^2} \) using Mathematica, Maple, or
Sage, you will obtain an expression involving erf (which stands for error function). The function erf(x) is defined as
\[
erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt
\]
so all three programs essentially rewrite the given integral in terms of a closely related integral. From one point of view this is what we did in integrating 1/x, since the natural logarithm function is (formally) defined as
\[\ln x = \int_1^x \frac{1}{t} dt\]
(see Section 5.10).

**Example 6** A particle moves along an x-axis in such a way that its velocity \(v(t)\) at time \(t\) is
\[v(t) = 30 \cos^7 t \sin^4 t \quad (t \geq 0)\]
Graph the position versus time curve for the particle, given that the particle is at \(x = 1\) when \(t = 0\).

**Solution.** Since \(dx/dt = v(t)\) and \(x = 1\) when \(t = 0\), the position function \(x(t)\) is given by
\[x(t) = 1 + \int_0^t v(s) \, ds\]
Some computer algebra systems will allow this expression to be entered directly into a command for plotting functions, but it is often more efficient to perform the integration first. The authors’ integration utility yields
\[x = \int 30 \cos^7 t \sin^4 t \, dt\]
\[= -\frac{30}{11} \sin^{11} t + 10 \sin^9 t - \frac{90}{7} \sin^7 t + 6 \sin^5 t + C\]
where we have added the required constant of integration. Using the initial condition \(x(0) = 1\), we substitute the values \(x = 1\) and \(t = 0\) into this equation to find that \(C = 1\), so
\[x(t) = -\frac{30}{11} \sin^{11} t + 10 \sin^9 t - \frac{90}{7} \sin^7 t + 6 \sin^5 t + 1 \quad (t \geq 0)\]
The graph of \(x\) versus \(t\) is shown in Figure 7.6.2.

**QUICK CHECK EXERCISES 7.6** (See page 533 for answers.)

1. Find an integral formula in the Endpaper Integral Table that can be used to evaluate the integral. Do not evaluate the integral.
   (a) \(\int \frac{2x}{3x + 4} \, dx\)  
   (b) \(\int \frac{1}{x^3 + 4} \, dx\)  
   (c) \(\int x \sqrt{3x + 2} \, dx\)  
   (d) \(\int x \ln x \, dx\)  
2. In each part, make the indicated \(u\)-substitution, and then find an integral formula in the Endpaper Integral Table that can be used to evaluate the integral. Do not evaluate the integral.
   (a) \(\int \frac{x}{1 + e^{2x}} \, dx; \ u = x^2\)  
   (b) \(\int e^{\sqrt{x}} \, dx; \ u = \sqrt{x}\)  
   (c) \(\int \frac{e^x}{1 + \sin(e^x)} \, dx; \ u = e^x\)  
   (d) \(\int \frac{1}{1 + 2x^2} \, dx; \ u = 2x\)  
3. In each part, use the Endpaper Integral Table to evaluate the integral. (If necessary, first make an appropriate substitution or complete the square.)
EXERCISE SET 7.6  CAS

1–24 (a) Use the Endpaper Integral Table to evaluate the given integral. (b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a).

1. \( \int \frac{4x}{3x-1} \, dx \)
2. \( \int \frac{x}{4-5x} \, dx \)
3. \( \int \frac{1}{x(2x+5)} \, dx \)
4. \( \int \frac{1}{x^2(1-5x)} \, dx \)
5. \( \int x\sqrt{2x+3} \, dx \)
6. \( \int \frac{x}{\sqrt{2-x}} \, dx \)
7. \( \int \frac{1}{x\sqrt{4-3x}} \, dx \)
8. \( \int \frac{1}{x\sqrt{3x-4}} \, dx \)
9. \( \int \frac{1}{16-x^2} \, dx \)
10. \( \int \frac{1}{x^2-9} \, dx \)
11. \( \int \sqrt{x^2-3} \, dx \)
12. \( \int \frac{\sqrt{x^2-5}}{x^2} \, dx \)
13. \( \int \frac{x^2}{\sqrt{x^4+4}} \, dx \)
14. \( \int \frac{1}{x^2\sqrt{x^2-2}} \, dx \)
15. \( \int \frac{\sqrt{9-x^2}}{x} \, dx \)
16. \( \int \frac{\sqrt{4-x^2}}{x} \, dx \)
17. \( \int \frac{\sqrt{4-x^2}}{x} \, dx \)
18. \( \int \frac{1}{x\sqrt{6x-x^2}} \, dx \)
19. \( \int \sin 3x \sin 4x \, dx \)
20. \( \int \sin 2x \cos 5x \, dx \)
21. \( \int x^3 \ln x \, dx \)
22. \( \int \frac{\ln x}{x^3} \, dx \)
23. \( \int e^{-2x} \sin 3x \, dx \)
24. \( \int e^x \cos 2x \, dx \)

25–36 (a) Make the indicated substitution, and then use the Endpaper Integral Table to evaluate the integral. (b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a).

25. \( \int e^{x^2} \left(\frac{1}{4-3e^{x^2}}\right) \, dx \), \( u = e^{x^2} \)
26. \( \int \frac{\sin 2x}{(\cos 2x)(-3-\cos 2x)} \, dx \), \( u = \cos 2x \)
27. \( \int \frac{1}{\sqrt{9x+4}} \, dx \), \( u = 3\sqrt{x} \)
28. \( \int \frac{\cos 4x}{9 + \sin^2 4x} \, dx \), \( u = \sin 4x \)
29. \( \int \frac{1}{\sqrt{4x^2-9}} \, dx \), \( u = 2x \)
30. \( \int x\sqrt{2x^4+3} \, dx \), \( u = \sqrt{2}x^2 \)
31. \( \int \frac{4x^5}{\sqrt{2-4x^2}} \, dx \), \( u = 2x^2 \)
32. \( \int \frac{1}{x^2\sqrt{3-4x^2}} \, dx \), \( u = 2x \)
33. \( \int \frac{\sin^2(\ln x)}{x} \, dx \), \( u = \ln x \)
34. \( \int e^{-2x} \cos^2(\sqrt{2}x) \, dx \), \( u = e^{-2x} \)
35. \( \int xe^{-2x} \, dx \), \( u = -2x \)
36. \( \int \ln(3x+1) \, dx \), \( u = 3x+1 \)

37–48 (a) Make an appropriate substitution, and then use the Endpaper Integral Table to evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a).

37. \( \int \frac{\cos 3x}{(\sin 3x)(\sin 3x+1)^2} \, dx \)
38. \( \int \frac{\ln x}{x\sqrt{4\ln x-1}} \, dx \)
39. \( \int \frac{x}{16x^4-1} \, dx \)
40. \( \int \frac{e^x}{3-4e^{2x}} \, dx \)
41. \( \int e\sqrt{3-4e^{2x}} \, dx \)
42. \( \int \frac{\sqrt{4-9x^2}}{x^2} \, dx \)
43. \( \int \sqrt{5x-9x^2} \, dx \)
44. \( \int \frac{1}{x\sqrt{x-5x^2}} \, dx \)
45. \( \int 4x \sin 2x \, dx \)
46. \( \int \cos \sqrt{x} \, dx \)
47. \( \int e^{-\sqrt{x}} \, dx \)
48. \( \int x \ln(2+x^2) \, dx \)

49–52 (a) Complete the square, make an appropriate substitution, and then use the Endpaper Integral Table to evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution or square completion), and then confirm that the result is equivalent to that in part (a).

49. \( \int \frac{1}{x^2+6x-7} \, dx \)
50. \( \int \sqrt{3-2x-x^2} \, dx \)
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51. \[ \int \frac{x}{\sqrt{5 + 4x - x^2}} \, dx \]
52. \[ \int \frac{x}{x^2 + 6x + 13} \, dx \]

C 53–64 (a) Make an appropriate \( u \)-substitution of the form \( u = x^{1/n} \) or \( u = (x + a)^{1/n} \), and then evaluate the integral.
(b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a).

53. \[ \int x \sqrt{x - 2} \, dx \]
54. \[ \int \frac{x}{\sqrt{x + 1}} \, dx \]
55. \[ \int x^5 \sqrt{x^3 + 1} \, dx \]
56. \[ \int \frac{1}{x \sqrt{x^3 - 1}} \, dx \]
57. \[ \int \frac{dx}{x - \sqrt{x}} \]
58. \[ \int \frac{dx}{\sqrt{x + \sqrt{x}} \sqrt{x}} \]
59. \[ \int \frac{dx}{x(1 - x^{1/4})} \]
60. \[ \int \frac{dx}{x + 1} \]
61. \[ \int \frac{dx}{x^{1/2} - x^{1/3}} \]
62. \[ \int \frac{dx}{1 + \sqrt{x}} \]
63. \[ \int \frac{x^3}{\sqrt{1 + x^2}} \, dx \]
64. \[ \int \frac{dx}{(x + 3)^{1/3}} \]

C 65–70 (a) Make \( u \)-substitution (5) to convert the integrand to a rational function of \( u \), and then evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a).

65. \[ \int \frac{dx}{1 + \sin x + \cos x} \]
66. \[ \int \frac{dx}{2 + \sin x} \]
67. \[ \int \frac{d\theta}{1 - \cos \theta} \]
68. \[ \int \frac{dx}{4 \sin x - 3 \cos x} \]
69. \[ \int \frac{dx}{\sin x + \tan x} \]
70. \[ \int \frac{dx}{\sin x + \tan x} \]

71–72 Use any method to solve for \( x \).

71. \[ \int_{1/2}^1 \frac{1}{t(4 - t)} \, dt = 0.5, \quad 2 < x < 4 \]
72. \[ \int_{1}^{\frac{1}{2}} \frac{1}{t\sqrt{7 - t}} \, dt = 1, \quad x > \frac{1}{2} \]

73–76 Use any method to find the area of the region enclosed by the curves.

73. \( y = \sqrt{25 - x^2}, \quad y = 0, \quad x = 0, \quad x = 4 \)
74. \( y = \sqrt{9x^2 - 4}, \quad y = 0, \quad x = 2 \)
75. \( y = \frac{1}{25 - 16x^2}, \quad y = 0, \quad x = 0, \quad x = 1 \)
76. \( y = \sqrt{x} \ln x, \quad y = 0, \quad x = 4 \)

77–80 Use any method to find the volume of the solid generated when the region enclosed by the curves is revolved about the \( y \)-axis.

77. \( y = \cos x, \quad y = 0, \quad x = 0, \quad x = \pi/2 \)
78. \( y = \sqrt{x - 4}, \quad y = 0, \quad x = 8 \)
79. \( y = e^{-x}, \quad y = 0, \quad x = 0, \quad x = 3 \)
80. \( y = \ln x, \quad y = 0, \quad x = 5 \)

81–82 Use any method to find the arc length of the curve.

81. \( y = 2x^2, \quad 0 \leq x \leq 2 \)
82. \( y = 3 \ln x, \quad 1 \leq x \leq 3 \)

83–84 Use any method to find the area of the surface generated by revolving the curve about the \( x \)-axis.

83. \( y = \sin x, \quad 0 \leq x \leq \pi \)
84. \( y = 1/x, \quad 1 \leq x \leq 4 \)

C 85–86 Information is given about the motion of a particle moving along a coordinate line.

(a) Use a CAS to find the position function of the particle for \( t \geq 0 \).
(b) Graph the position versus time curve.

85. \( \dot{v}(t) = 20 \cos^3 t \sin^3 t, \quad v(0) = 2 \)
86. \( a(t) = e^{-t} \sin 2t \sin 4t, \quad v(0) = 0, \quad s(0) = 10 \)

FOCUS ON CONCEPTS

87. (a) Use the substitution \( u = \tan(x/2) \) to show that
\[ \int \sec x \, dx = \ln \left| \frac{1 + \tan(x/2)}{1 - \tan(x/2)} \right| + C \]
and confirm that this is consistent with Formula (22) of Section 7.3.
(b) Use the result in part (a) to show that
\[ \int \sec x \, dx = \ln \left| \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) \right| + C \]

88. Use the substitution \( u = \tan(x/2) \) to show that
\[ \int \csc x \, dx = \frac{1}{2} \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| + C \]
and confirm that this is consistent with the result in Exercise 65(a) of Section 7.3.

89. Find a substitution that can be used to integrate rational functions of \( \sin x \) and \( \cosh x \) and use your substitution to evaluate
\[ \int \frac{dx}{2 \cosh x + \sinh x} \]
without expressing the integrand in terms of \( e^x \) and \( e^{-x} \).

C 90–93 Some integrals that can be evaluated by hand cannot be evaluated by all computer algebra systems. Evaluate the integral by hand, and determine if it can be evaluated on your CAS.

90. \[ \int \frac{x^3}{\sqrt{1 - x^4}} \, dx \]
91. \[ \int (\cos^{32} x \sin^{30} x - \cos^{30} x \sin^{32} x) \, dx \]
92. \[ \int \sqrt{x - \sqrt{x^2 - 4}} \, dx \] [Hint: \( \frac{1}{2}(\sqrt{x+2} - \sqrt{x-2})^2 = ? \)]
93. \[ \int \frac{1}{x^{10} + x} \, dx \]

[Hint: Rewrite the denominator as \( x^{10}(1 + x^{-9}) \).]

94. Let

\[ f(x) = \frac{-2x^5 + 26x^4 + 15x^3 + 6x^2 + 20x + 43}{x^6 - x^5 - 18x^4 - 2x^3 - 39x^2 - x - 20} \]

\( \text{C} \)

\( f(x) \)

(a) Use a CAS to factor the denominator, and then write down the form of the partial fraction decomposition. You need not find the values of the constants.

(b) Check your answer in part (a) by using the CAS to find the partial fraction decomposition of \( f \).

(c) Integrate \( f \) by hand, and then check your answer by integrating with the CAS.

\[ \text{QUICK CHECK ANSWERS 7.6} \]

1. (a) Formula (60) (b) Formula (108) (c) Formula (102) (d) Formula (50)

2. (a) Formula (25) (b) Formula (51)

(c) Formula (18) (d) Formula (97)

3. (a) \( \frac{1}{4} \ln \left| \frac{x + 2}{x - 2} \right| + C \) (b) \( \frac{1}{6} \sin 3x + \frac{1}{2} \sin x + C \) (c) \( -\frac{e^x}{2\sqrt{1 - e^{2x}}} + \frac{1}{2} \sin^{-1} e^x + C \)

(d) \( \frac{1}{2} \ln (x^2 - 4x + 8) + \tan^{-1} \frac{x - 2}{2} + C \)

\[ \text{7.7 NUMERICAL INTEGRATION; SIMPSON'S RULE} \]

If it is necessary to evaluate a definite integral of a function for which an antiderivative cannot be found, then one must settle for some kind of numerical approximation of the integral. In Section 5.4 we considered three such approximations in the context of areas—left endpoint approximation, right endpoint approximation, and midpoint approximation. In this section we will extend those methods to general definite integrals, and we will develop some new methods that often provide more accuracy with less computation. We will also discuss the errors that arise in integral approximations.

\[ \text{A REVIEW OF RIEMANN SUM APPROXIMATIONS} \]

Recall from Section 5.5 that the definite integral of a continuous function \( f \) over an interval \([a, b]\) may be computed as

\[ \int_{a}^{b} f(x) \, dx = \lim_{\Delta x_{k} \to 0} \sum_{k=1}^{n} f(x^*_k) \Delta x_k \]

where the sum that appears on the right side is called a Riemann sum. In this formula, \( \Delta x_k \) is the width of the \( k \)th subinterval of a partition \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \) of \([a, b]\) into \( n \) subintervals, and \( x^*_k \) denotes an arbitrary point in the \( k \)th subinterval. If we take all subintervals of the same width, so that \( \Delta x_k = (b - a)/n \), then as \( n \) increases the Riemann sum will eventually be a good approximation to the definite integral. We denote this by writing

\[ \int_{a}^{b} f(x) \, dx \approx \left( \frac{b - a}{n} \right) \left[ f(x^*_1) + f(x^*_2) + \cdots + f(x^*_n) \right] \] (1)

If we denote the values of \( f \) at the endpoints of the subintervals by

\[ y_0 = f(a), \quad y_1 = f(x_1), \quad y_2 = f(x_2), \ldots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(x_n) \]

and the values of \( f \) at the midpoints of the subintervals by

\[ y_{m_1}, y_{m_2}, \ldots, y_{m_n} \]

then it follows from (1) that the left endpoint, right endpoint, and midpoint approximations discussed in Section 5.4 can be expressed as shown in Table 7.7.1. Although we originally
Table 7.7.1

<table>
<thead>
<tr>
<th>LEFT ENDPOINT APPROXIMATION</th>
<th>RIGHT ENDPOINT APPROXIMATION</th>
<th>MIDPOINT APPROXIMATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int_a^b f(x) , dx \approx \frac{b-a}{n} \left[ y_0 + y_1 + \cdots + y_{n-1} \right] )</td>
<td>( \int_a^b f(x) , dx \approx \frac{b-a}{n} \left[ y_1 + y_2 + \cdots + y_n \right] )</td>
<td>( \int_a^b f(x) , dx \approx \frac{b-a}{n} \left[ y_{m_1} + y_{m_2} + \cdots + y_{m_n} \right] )</td>
</tr>
</tbody>
</table>

TRAPEZOIDAL APPROXIMATION

It will be convenient in this section to denote the left endpoint, right endpoint, and midpoint approximations with \( L_n, R_n, \) and \( M_n \), respectively. Of the three approximations, the midpoint approximation is most widely used in applications. If we take the average of \( L_n \) and \( R_n \), then we obtain another important approximation denoted by

\[
T_n = \frac{1}{2}(L_n + R_n)
\]

called the trapezoidal approximation:

\[\int_a^b f(x) \, dx \approx T_n = \frac{b-a}{2n} \left[ y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n \right] \quad (2)\]

The name “trapezoidal approximation” results from the fact that in the case where \( f \) is nonnegative on the interval of integration, the approximation \( T_n \) is the sum of the trapezoidal areas shown in Figure 7.7.1 (see Exercise 51).

Example 1

In Table 7.7.2 we have approximated

\[\ln 2 = \int_1^2 \frac{1}{x} \, dx\]

using the midpoint approximation and the trapezoidal approximation.\(^1\) In each case we used \( n = 10 \) subdivisions of the interval \([1, 2]\), so that

\[
\frac{b-a}{n} = \frac{2-1}{10} = 0.1 \quad \text{Midpoint} \quad \text{and} \quad \frac{b-a}{2n} = \frac{2-1}{20} = 0.05 \quad \text{Trapezoidal}
\]

\(^1\)Throughout this section we will show numerical values to nine places to the right of the decimal point. If your calculating utility does not show this many places, then you will need to make the appropriate adjustments. What is important here is that you understand the principles being discussed.
7.7 Numerical Integration; Simpson’s Rule

Table 7.7.2

<table>
<thead>
<tr>
<th>MIDPOINT APPROXIMATION</th>
<th>TRAPEZOIDAL APPROXIMATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>m_i</td>
</tr>
<tr>
<td>1</td>
<td>1.05</td>
</tr>
<tr>
<td>2</td>
<td>1.15</td>
</tr>
<tr>
<td>3</td>
<td>1.25</td>
</tr>
<tr>
<td>4</td>
<td>1.35</td>
</tr>
<tr>
<td>5</td>
<td>1.45</td>
</tr>
<tr>
<td>6</td>
<td>1.55</td>
</tr>
<tr>
<td>7</td>
<td>1.65</td>
</tr>
<tr>
<td>8</td>
<td>1.75</td>
</tr>
<tr>
<td>9</td>
<td>1.85</td>
</tr>
<tr>
<td>10</td>
<td>1.95</td>
</tr>
<tr>
<td></td>
<td>6.928353603</td>
</tr>
</tbody>
</table>

\[ \int_{1}^{2} \frac{1}{x} \, dx = (0.1)(6.928353603) = 0.692835360 \]

\[ \int_{1}^{2} \frac{1}{x} \, dx = (0.05)(13.875428063) = 0.693771403 \]

**COMPARISON OF THE MIDPOINT AND TRAPEZOIDAL APPROXIMATIONS**

We define the **errors** in the midpoint and trapezoidal approximations to be

\[ E_M = \int_a^b f(x) \, dx - M_n \quad \text{and} \quad E_T = \int_a^b f(x) \, dx - T_n \]  

(3–4)

respectively, and we define \(|E_M|\) and \(|E_T|\) to be the **absolute errors** in these approximations. The absolute errors are nonnegative and do not distinguish between underestimates and overestimates.

**Example 2** The value of \(\ln 2\) to nine decimal places is

\[ \ln 2 = \int_{1}^{2} \frac{1}{x} \, dx \approx 0.693147181 \]  

(5)

so we see from Tables 7.7.2 and 7.7.3 that the absolute errors in approximating \(\ln 2\) by \(M_{10}\) and \(T_{10}\) are

\[ |E_M| = |\ln 2 - M_{10}| \approx 0.000311821 \]

\[ |E_T| = |\ln 2 - T_{10}| \approx 0.000624222 \]

Thus, the midpoint approximation is more accurate than the trapezoidal approximation in this case.

Table 7.7.3

<table>
<thead>
<tr>
<th>NINE DECIMAL PLACES</th>
<th>APPROXIMATION</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.693147181</td>
<td>(M_{10} = 0.692835360)</td>
<td>(E_M = \ln 2 - M_{10} = 0.000311821)</td>
</tr>
<tr>
<td>0.693147181</td>
<td>(T_{10} = 0.693771403)</td>
<td>(E_T = \ln 2 - T_{10} = -0.000624222)</td>
</tr>
</tbody>
</table>
Chapter 7 / Principles of Integral Evaluation

It is not accidental in Example 2 that the midpoint approximation of \( \ln 2 \) was more accurate than the trapezoidal approximation. To see why this is so, we first need to look at the midpoint approximation from another point of view. To simplify our explanation, we will assume that \( f \) is nonnegative on \([a, b]\), though the conclusions we reach will be true without this assumption.

If \( f \) is a differentiable function, then the midpoint approximation is sometimes called the **tangent line approximation** because for each subinterval of \([a, b]\) the area of the rectangle used in the midpoint approximation is equal to the area of the trapezoid whose upper boundary is the tangent line to \( y = f(x) \) at the midpoint of the subinterval (Figure 7.7.2). The equality of these areas follows from the fact that the shaded areas in the figure are congruent. We will now show how this point of view about midpoint approximations can be used to establish useful criteria for determining which of \( M_n \) or \( T_n \) produces the better approximation of a given integral.

In Figure 7.7.3a we have isolated a subinterval of \([a, b]\) on which the graph of a function \( f \) is concave down, and we have shaded the areas that represent the errors in the midpoint and trapezoidal approximations over the subinterval. In Figure 7.7.3b we show a succession of four illustrations which make it evident that the error from the midpoint approximation is less than that from the trapezoidal approximation. If the graph of \( f \) were concave up, analogous figures would lead to the same conclusion. (This argument, due to Frank Buck, appeared in *The College Mathematics Journal*, Vol. 16, No. 1, 1985.)

Figure 7.7.3a also suggests that on a subinterval where the graph is concave down, the midpoint approximation is larger than the value of the integral and the trapezoidal approximation is smaller. On an interval where the graph is concave up it is the other way around. In summary, we have the following result, which we state without formal proof:

**7.7.1 THEOREM** Let \( f \) be continuous on \([a, b]\), and let \(|E_M|\) and \(|E_T|\) be the absolute errors that result from the midpoint and trapezoidal approximations of \( \int_a^b f(x) \, dx \) using \( n \) subintervals.

(a) If the graph of \( f \) is either concave up or concave down on \((a, b)\), then \(|E_M| < |E_T|\), that is, the absolute error from the midpoint approximation is less than that from the trapezoidal approximation.

(b) If the graph of \( f \) is concave down on \((a, b)\), then

\[
T_n < \int_a^b f(x) \, dx < M_n
\]

(c) If the graph of \( f \) is concave up on \((a, b)\), then

\[
M_n < \int_a^b f(x) \, dx < T_n
\]
### 7.7 Numerical Integration; Simpson’s Rule

**Example 3** Since the graph of \( f(x) = \frac{1}{x} \) is continuous on the interval \([1, 2]\) and concave up on the interval \((1, 2)\), it follows from part (a) of Theorem 7.7.1 that \( M_n \) will always provide a better approximation than \( T_n \) for

\[
\int_1^2 \frac{1}{x} \, dx = \ln 2
\]

Moreover, if follows from part (c) of Theorem 7.7.1 that \( M_n < \ln 2 < T_n \) for every positive integer \( n \). Note that this is consistent with our computations in Example 2. ▶

**Example 4** The midpoint and trapezoidal approximations can be used to approximate \( \sin 1 \) by using the integral

\[
\sin 1 = \int_0^1 \cos x \, dx
\]

Since \( f(x) = \cos x \) is continuous on \([0, 1]\) and concave down on \((0, 1)\), it follows from parts (a) and (b) of Theorem 7.7.1 that the absolute error in \( M_n \) will be less than that in \( T_n \), and that \( T_n < \sin 1 < M_n \) for every positive integer \( n \). This is consistent with the results in Table 7.7.4 for \( n = 5 \) (intermediate computations are omitted). ▶

#### Table 7.7.4

<table>
<thead>
<tr>
<th>sin 1 (NINE DECIMAL PLACES)</th>
<th>APPROXIMATION</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.841470985</td>
<td>M₅ ≈ 0.842875074</td>
<td>( E_M = \sin 1 - M_5 = -0.001404089 )</td>
</tr>
<tr>
<td>0.841470985</td>
<td>T₅ ≈ 0.838664210</td>
<td>( E_T = \sin 1 - T_5 = 0.002806775 )</td>
</tr>
</tbody>
</table>

**Example 5** Table 7.7.5 shows approximations for \( \sin 3 = \int_0^1 \cos x \, dx \) using the midpoint and trapezoidal approximations with \( n = 10 \) subdivisions of the interval \([0, 3]\). Note that \( |E_M| < |E_T| \) and \( T_{10} < \sin 3 < M_{10} \), although these results are not guaranteed by Theorem 7.7.1 since \( f(x) = \cos x \) changes concavity on the interval \([0, 3]\). ▶

#### Table 7.7.5

<table>
<thead>
<tr>
<th>sin 3 (NINE DECIMAL PLACES)</th>
<th>APPROXIMATION</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.141120008</td>
<td>M₁₀ ≈ 0.141650601</td>
<td>( E_M = \sin 3 - M_{10} = -0.000530592 )</td>
</tr>
<tr>
<td>0.141120008</td>
<td>T₁₀ ≈ 0.140060017</td>
<td>( E_T = \sin 3 - T_{10} = 0.001059991 )</td>
</tr>
</tbody>
</table>

### Simpson’s Rule

When the left and right endpoint approximations are averaged to produce the trapezoidal approximation, a better approximation often results. We now see how a weighted average of the midpoint and trapezoidal approximations can yield an even better approximation.

The numerical evidence in Tables 7.7.3, 7.7.4, and 7.7.5 reveals that \( E_T \approx -2E_M \), so that \( 2E_M + E_T \approx 0 \) in these instances. This suggests that

\[
3 \int_a^b f(x) \, dx = 2 \int_a^b f(x) \, dx + \int_a^b f(x) \, dx
\]

\[
= 2(M_k + E_M) + (T_k + E_T)
\]

\[
= (2M_k + T_k) + (2E_M + E_T)
\]

\[
\approx 2M_k + T_k
\]
WARNING

Note that in (7) the subscript $n$ in $S_n$ is always even since it is twice the value of the subscripts for the corresponding midpoint and trapezoidal approximations. For example,

$$S_{10} = \frac{1}{2}(2M_5 + T_5)$$

and

$$S_{20} = \frac{1}{2}(2M_{10} + T_{10})$$

This gives

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(2M_k + T_k)$$

(6)

The midpoint approximation $M_k$ in (6) requires the evaluation of $f$ at $k$ points in the interval $[a, b]$, and the trapezoidal approximation $T_k$ in (6) requires the evaluation of $f$ at $k + 1$ points in $[a, b]$. Thus, $\frac{1}{2}(2M_k + T_k)$ uses $2k + 1$ values of $f$, taken at equally spaced points in the interval $[a, b]$. These points are obtained by partitioning $[a, b]$ into $2k$ equal subintervals indicated by the left endpoints, right endpoints, and midpoints used in $T_k$ and $M_k$, respectively. Setting $n = 2k$, we use $S_n$ to denote the weighted average of $M_k$ and $T_k$ in (6). That is,

$$S_n = S_{2k} = \frac{1}{2}(2M_k + T_k) \quad \text{or} \quad S_n = \frac{1}{4}(2M_{n/2} + T_{n/2})$$

(7)

Table 7.7.6 displays the approximations $S_n$ corresponding to the data in Tables 7.7.3 to 7.7.5.

<table>
<thead>
<tr>
<th>Function Value (Nine Decimal Places)</th>
<th>Approximation</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln 2 = 0.693147181$</td>
<td>$\int_0^\ln 2 (1/x),dx = S_{20} = \frac{1}{2}(2M_{10} + T_{10}) = 0.693147375$</td>
<td>$-0.000000194$</td>
</tr>
<tr>
<td>$\sin 1 = 0.841470985$</td>
<td>$\int_0^\sin 1 \cos x,dx = S_{10} = \frac{1}{2}(2M_5 + T_5) = 0.841471453$</td>
<td>$-0.000000468$</td>
</tr>
<tr>
<td>$\sin 3 = 0.141120008$</td>
<td>$\int_0^\sin 3 \cos x,dx = S_{20} = \frac{1}{2}(2M_{10} + T_{10}) = 0.141120406$</td>
<td>$-0.000000398$</td>
</tr>
</tbody>
</table>

Using the midpoint approximation formula in Table 7.7.1 and Formula (2) for the trapezoidal approximation, we can derive a similar formula for $S_n$. We start by partitioning the interval $[a, b]$ into an even number of equal subintervals. If $n$ is the number of subintervals, then each subinterval has length $(b - a)/n$. Label the endpoints of these subintervals successively by $a = x_0, x_1, x_2, \ldots, x_n = b$. Then $x_0, x_2, x_4, \ldots, x_n$ define a partition of $[a, b]$ into $n/2$ equal intervals, each of length $2(b - a)/n$, and the midpoints of these subintervals are $x_1, x_3, x_5, \ldots, x_{n-1}$, respectively, as illustrated in Figure 7.7.4. Using $y_i = f(x_i)$, we have

$$2M_{n/2} = 2 \left( \frac{2(b - a)}{n} \right) [y_1 + y_3 + \cdots + y_{n-1}]$$

$$= \left( \frac{b - a}{n} \right) [4y_1 + 4y_3 + \cdots + 4y_{n-1}]$$

Noting that $(b - a)/(2(n/2)) = (b - a)/n$, we can express $T_{n/2}$ as

$$T_{n/2} = \left( \frac{b - a}{n} \right) [y_0 + 2y_2 + 2y_4 + \cdots + 2y_{n-2} + y_n]$$

Thus, $S_n = \frac{1}{2}(2M_{n/2} + T_{n/2})$ can be expressed as

$$S_n = \frac{1}{3} \left( \frac{b - a}{n} \right) [y_0 + 4y_1 + 2y_2 + 2y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n]$$

(8)

The approximation

$$\int_a^b f(x)\,dx \approx S_n$$

(9)

with $S_n$ as given in (8) is known as Simpson’s rule. We denote the error in this approximation by

$$E_S = \int_a^b f(x)\,dx - S_n$$

(10)

As before, the absolute error in the approximation (9) is given by $|E_S|$. 

### Example 6

In Table 7.7.7 we have used Simpson’s rule with \( n = 10 \) subintervals to obtain the approximation

\[
\ln 2 = \int_1^2 \frac{1}{x} \, dx \approx S_{10} = 0.693150231
\]

For this approximation,

\[
\frac{1}{3} \left( \frac{6}{n} \right) = \frac{1}{3} \left( \frac{2 - 1}{10} \right) = \frac{1}{30}
\]

<table>
<thead>
<tr>
<th>Table 7.7.7</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AN APPROXIMATION TO ∫(2) 1/x USING SIMPSON’S RULE</strong></td>
</tr>
<tr>
<td>( i )</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
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<td>3</td>
</tr>
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<td>7</td>
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<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

\[
\int_1^2 \frac{1}{x} \, dx = \left( \frac{1}{30} \right) (20.794506921) = 0.693150231
\]

Thomas Simpson (1710–1761) English mathematician. Simpson was the son of a weaver. He was trained to follow in his father’s footsteps and had little formal education in his early life. His interest in science and mathematics was aroused in 1724, when he witnessed an eclipse of the Sun and received two books from a peddler, one on astrology and the other on arithmetic. Simpson quickly absorbed their contents and soon became a successful local fortune teller. His improved financial situation enabled him to give up weaving and marry his landlady. Then in 1733 some mysterious “fortunate incident” forced him to move. He settled in Derby, where he taught in an evening school and worked at weaving during the day. In 1736 he moved to London and published his first mathematical work in a periodical called the Ladies’ Diary (of which he later became the editor). In 1737 he published a successful calculus textbook that enabled him to give up weaving completely and concentrate on textbook writing and teaching. His fortunes improved further in 1740 when one Robert Heath accused him of plagiarism. The publicity was marvelous, and Simpson proceeded to dash off a succession of best-selling textbooks: *Algebra* (ten editions plus translations), *Geometry* (twelve editions plus translations), *Trigonometry* (five editions plus translations), and numerous others. It is interesting to note that Simpson did not discover the rule that bears his name—it was a well-known result by Simpson’s time.

[Image: http://www-history.mcs.st-and.ac.uk/Posters/820.html]
Although $S_{10}$ is a weighted average of $M_{10}$ and $T_{10}$, it makes sense to compare $S_{10}$ to $M_{10}$ and $T_{10}$, since the sums for these three approximations involve the same number of terms. Using the values for $M_{10}$ and $T_{10}$ from Example 2 and the value for $S_{10}$ in Table 7.7.7, we have

$$|E_M| = |\ln 2 - M_{10}| \approx |0.693147181 - 0.692835360| = 0.000311821$$

$$|E_T| = |\ln 2 - T_{10}| \approx |0.693147181 - 0.693771403| = 0.000624222$$

$$|E_S| = |\ln 2 - S_{10}| \approx |0.693147181 - 0.693150231| = 0.000003050$$

Comparing these absolute errors, it is clear that $S_{10}$ is a much more accurate approximation of $\ln 2$ than either $M_{10}$ or $T_{10}$.

GEOMETRIC INTERPRETATION OF SIMPSON’S RULE

The midpoint (or tangent line) approximation and the trapezoidal approximation for a definite integral are based on approximating a segment of the curve $y = f(x)$ by line segments. Intuition suggests that we might improve on these approximations using parabolic arcs rather than line segments, thereby accounting for concavity of the curve $y = f(x)$ more closely.

At the heart of this idea is a formula, sometimes called the one-third rule. The one-third rule expresses a definite integral of a quadratic function $g(x) = Ax^2 + Bx + C$ in terms of the values $Y_0, Y_1, Y_2$ of $g$ at the left endpoint, midpoint, and right endpoint, respectively, of the interval of integration $[m - \Delta x, m + \Delta x]$ (see Figure 7.7.5):

$$\int_{m-\Delta x}^{m+\Delta x} (Ax^2 + Bx + C) dx = \frac{\Delta x}{3} [Y_0 + 4Y_1 + Y_2]$$

Verification of the one-third rule is left for the reader (Exercise 53). By applying the one-third rule to subintervals $[x_{2k-2}, x_{2k}, k = 1, \ldots, n/2$, one arrives at Formula (8) for Simpson’s rule (Exercise 54). Thus, Simpson’s rule corresponds to the integral of a piecewise-quadratic approximation to $f(x)$.

ERROR BOUNDS

With all the methods studied in this section, there are two sources of error: the intrinsic or truncation error due to the approximation formula, and the roundoff error introduced in the calculations. In general, increasing $n$ reduces the truncation error but increases the roundoff error, since more computations are required for larger $n$. In practical applications, it is important to know how large $n$ must be taken to ensure that a specified degree of accuracy is obtained. The analysis of roundoff error is complicated and will not be considered here. However, the following theorems, which are proved in books on numerical analysis, provide upper bounds on the truncation errors in the midpoint, trapezoidal, and Simpson’s rule approximations.

**THEOREM (Midpoint and Trapezoidal Error Bounds)** If $f''$ is continuous on $[a, b]$ and if $K_2$ is the maximum value of $|f''(x)|$ on $[a, b]$, then

(a) $|E_M| = \left| \int_a^b f(x) \, dx - M_n \right| \leq \frac{(b - a)^3 K_2}{24n^2}$

(b) $|E_T| = \left| \int_a^b f(x) \, dx - T_n \right| \leq \frac{(b - a)^3 K_2}{12n^2}$
7.7 Numerical Integration; Simpson’s Rule

7.7.3 **Theorem (Simpson Error Bound)**  If \( f^{(4)} \) is continuous on \([a, b]\) and if \( K_4 \) is the maximum value of \(|f^{(4)}(x)|\) on \([a, b]\), then

\[
|E_S| = \left| \int_a^b f(x) \, dx - S_n \right| \leq \frac{(b - a)^5 K_4}{180 n^4} \quad (14)
\]

**Example 7**  Find an upper bound on the absolute error that results from approximating

\[
\ln 2 = \int_1^2 \frac{1}{x} \, dx
\]

using (a) the midpoint approximation \( M_{10} \), (b) the trapezoidal approximation \( T_{10} \), and (c) Simpson’s rule \( S_{10} \), each with \( n = 10 \) subintervals.

**Solution.**  We will apply Formulas (12), (13), and (14) with

\[
f(x) = \frac{1}{x}, \quad a = 1, \quad b = 2, \quad \text{and} \quad n = 10
\]

We have

\[
 f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{6}{x^4}, \quad f^{(4)}(x) = \frac{24}{x^5}
\]

Thus,

\[
|f''(x)| = \left| \frac{2}{x^3} \right| = \frac{2}{x^3}, \quad |f^{(4)}(x)| = \left| \frac{24}{x^5} \right| = \frac{24}{x^5}
\]

where we have dropped the absolute values because \( f''(x) \) and \( f^{(4)}(x) \) have positive values for \( 1 \leq x \leq 2 \). Since \( |f''(x)| \) and \( |f^{(4)}(x)| \) are continuous and decreasing on \([1, 2]\), both functions have their maximum values at \( x = 1 \); for \( |f''(x)| \) this maximum value is 2 and for \( |f^{(4)}(x)| \) the maximum value is 24. Thus we can take \( K_2 = 2 \) in (12) and (13), and \( K_4 = 24 \) in (14). This yields

\[
|E_M| \leq \frac{(b - a)^3 K_2}{24 n^2} = \frac{1^3 \cdot 2}{24 \cdot 10^2} \approx 0.000833333
\]

\[
|E_T| \leq \frac{(b - a)^3 K_2}{12 n^2} = \frac{1^3 \cdot 2}{12 \cdot 10^2} \approx 0.001666667
\]

\[
|E_S| \leq \frac{(b - a)^5 K_4}{180 n^4} = \frac{1^5 \cdot 24}{180 \cdot 10^4} \approx 0.000013333
\]

**Example 8**  How many subintervals should be used in approximating

\[
\ln 2 = \int_1^2 \frac{1}{x} \, dx
\]

by Simpson’s rule for five decimal-place accuracy?

**Solution.**  To obtain five decimal-place accuracy, we must choose the number of subintervals so that

\[
|E_S| \leq 0.000005 = 5 \times 10^{-6}
\]

From (14), this can be achieved by taking \( n \) in Simpson’s rule to satisfy

\[
\frac{(b - a)^5 K_4}{180 n^4} \leq 5 \times 10^{-6}
\]
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Taking $a = 1, b = 2$, and $K_4 = 24$ (found in Example 7) in this inequality yields

$$\frac{1^5 \cdot 24}{180 \cdot n^4} \leq 5 \times 10^{-6}$$

which, on taking reciprocals, can be rewritten as

$$n^4 \geq \frac{2 \times 10^6}{75} = \frac{8 \times 10^4}{3}$$

Thus,

$$n \geq \frac{20}{\sqrt[4]{6}} \approx 12.779$$

Since $n$ must be an even integer, the smallest value of $n$ that satisfies this requirement is $n = 14$. Thus, the approximation $S_{14}$ using 14 subintervals will produce five decimal-place accuracy.

**Remark**

In cases where it is difficult to find the values of $K_2$ and $K_4$ in Formulas (12), (13), and (14), these constants may be replaced by any larger constants. For example, suppose that a constant $K$ can be easily found with the certainty that $|f''(x)| < K$ on the interval. Then

$$|E_T| \leq \frac{(b-a)^3 K_2}{12n^2} \leq \frac{(b-a)^3 K}{12n^2}$$

and

$$|E_T| \leq \frac{1}{12n^2}$$

so the right side of (15) is also an upper bound on the value of $|E_T|$. Using $K$, however, will likely increase the computed value of $n$ needed for a given error tolerance. Many applications involve the resolution of competing practical issues, illustrated here through the trade-off between the convenience of finding a crude bound for $|f''(x)|$ versus the efficiency of using the smallest possible $n$ for a desired accuracy.

**Example 9**

How many subintervals should be used in approximating $\int_0^1 \cos(x^2) \, dx$ by the midpoint approximation for three decimal-place accuracy?

**Solution.** To obtain three decimal-place accuracy, we must choose $n$ so that

$$|E_M| \leq 0.0005 = 5 \times 10^{-4}$$

From (12) with $f(x) = \cos(x^2), a = 0$, and $b = 1$, an upper bound on $|E_M|$ is given by

$$|E_M| \leq \frac{K_2}{24n^2}$$

where $|K_2|$ is the maximum value of $|f''(x)|$ on the interval $[0, 1]$. However,

$$f'(x) = -2x \sin(x^2)$$

and

$$f''(x) = -4x^2 \cos(x^2) - 2 \sin(x^2) = -[4x^2 \cos(x^2) + 2 \sin(x^2)]$$

so that

$$|f''(x)| = |4x^2 \cos(x^2) + 2 \sin(x^2)|$$

It would be tedious to look for the maximum value of this function on the interval $[0, 1]$. For $x$ in $[0, 1]$, it is easy to see that each of the expressions $x^2, \cos(x^2)$, and $\sin(x^2)$ is bounded in absolute value by 1, so $|4x^2 \cos(x^2) + 2 \sin(x^2)| \leq 4 + 2 = 6$ on $[0, 1]$. We can improve on this by using a graphing utility to sketch $|f''(x)|$, as shown in Figure 7.7.6. It is evident from the graph that

$$|f''(x)| < 4 \quad \text{for} \quad 0 \leq x \leq 1$$
Thus, it follows from (17) that
\[
|E_M| \leq \frac{K_2}{24n^2} \leq \frac{4}{24n^2} = \frac{1}{6n^2}
\]
and hence we can satisfy (16) by choosing \( n \) so that
\[
\frac{1}{6n^2} \leq 5 \times 10^{-4}
\]
which, on taking reciprocals, can be written as
\[
n^2 > \frac{10^4}{30} \quad \text{or} \quad n > \frac{10^2}{\sqrt{30}} \approx 18.257
\]
The smallest integer value of \( n \) satisfying this inequality is \( n = 19 \). Thus, the midpoint approximation \( M_{19} \) using 19 subintervals will produce three decimal-place accuracy.

A COMPARISON OF THE THREE METHODS
Of the three methods studied in this section, Simpson’s rule generally produces more accurate results than the midpoint or trapezoidal approximations for the same amount of work. To make this plausible, let us express (12), (13), and (14) in terms of the subinterval width \( \Delta x = \frac{b - a}{n} \).

We obtain
\[
|E_M| \leq \frac{1}{24} K_2 (b - a)(\Delta x)^2 \quad (19)
\]
\[
|E_T| \leq \frac{1}{12} K_2 (b - a)(\Delta x)^2 \quad (20)
\]
\[
|E_S| \leq \frac{1}{180} K_4 (b - a)(\Delta x)^4 \quad (21)
\]
(verify). For Simpson’s rule, the upper bound on the absolute error is proportional to \((\Delta x)^4\), whereas the upper bound on the absolute error for the midpoint and trapezoidal approximations is proportional to \((\Delta x)^2\). Thus, reducing the interval width by a factor of 10, for example, reduces the error bound by a factor of 100 for the midpoint and trapezoidal approximations but reduces the error bound by a factor of 10,000 for Simpson’s rule. This suggests that, as \( n \) increases, the accuracy of Simpson’s rule improves much more rapidly than that of the other approximations.

As a final note, observe that if \( f(x) \) is a polynomial of degree 3 or less, then we have \( f^{(4)}(x) = 0 \) for all \( x \), so \( K_4 = 0 \) in (14) and consequently \( |E_S| = 0 \). Thus, Simpson’s rule gives exact results for polynomials of degree 3 or less. Similarly, the midpoint and trapezoidal approximations give exact results for polynomials of degree 1 or less. (You should also be able to see that this is so geometrically.)

Quick Check Exercises 7.7

1. Let \( T_n \) be the trapezoidal approximation for the definite integral of \( f(x) \) over an interval \([a, b]\) using \( n \) subintervals.
   (a) Expressed in terms of \( L_n \) and \( R_n \) (the left and right endpoint approximations), \( T_n = \) ________.
   (b) Expressed in terms of the function values \( y_0, y_1, \ldots, y_n \) at the endpoints of the subintervals, \( T_n = \) ________.

2. Let \( I \) denote the definite integral of \( f \) over an interval \([a, b]\) with \( T_n \) and \( M_n \) the respective trapezoidal and midpoint approximations of \( I \) for a given \( n \). Assume that the graph of \( f \) is concave up on the interval \([a, b]\) and order the quantities \( T_n, M_n, I \) from smallest to largest: ________ < ________ < ________.
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3. Let $S_n$ be the Simpson’s rule approximation for $\int_a^b f(x) \, dx$ using $n = 6$ subintervals.
   (a) Expressed in terms of $M_3$ and $T_3$ (the midpoint and trapezoidal approximations), $S_6 =$ ________
   (b) Using the function values $y_0, y_1, y_2, \ldots, y_6$ at the endpoints of the subintervals, $S_6 =$ ________

4. Assume that $f^{(4)}$ is continuous on $[0, 1]$ and that $f^{(k)}(x)$ satisfies $|f^{(k)}(x)| \leq 1$ on $[0, 1], k = 1, 2, 3, 4$. Find an upper bound on the absolute error that results from approximating the integral of $f$ over $[0, 1]$ using (a) the midpoint approximation $M_{10}$; (b) the trapezoidal approximation $T_{10}$; and (c) Simpson’s rule $S_{10}$.

5. Approximate $\int_1^3 \frac{1}{x^2} \, dx$ using the indicated method.
   (a) $M_1 =$ ________
   (b) $T_1 =$ ________
   (c) $S_2 =$ ________

EXERCISE SET 7.7 CAS

1–6 Approximate the integral using (a) the midpoint approximation $M_{10}$, (b) the trapezoidal approximation $T_{10}$, and (c) Simpson’s rule approximation $S_{20}$ using Formula (7). In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places.

1. $\int_0^3 \sqrt{x} + 1 \, dx$  
2. $\int_1^9 \frac{1}{\sqrt{x}} \, dx$  
3. $\int_0^{\pi/2} \cos x \, dx$

4. $\int_0^2 \sin x \, dx$  
5. $\int_1^3 e^{-2x} \, dx$  
6. $\int_0^3 \frac{1}{3x + 1} \, dx$

7–12 Use inequalities (12), (13), and (14) to find upper bounds on the errors in parts (a), (b), and (c) of the indicated exercise.

7. Exercise 1  
8. Exercise 2  
9. Exercise 3

10. Exercise 4  
11. Exercise 5  
12. Exercise 6

13–18 Use inequalities (12), (13), and (14) to find a number $n$ of subintervals for (a) the midpoint approximation $M_n$, (b) the trapezoidal approximation $T_n$, and (c) Simpson’s rule approximation $S_n$ to ensure that the absolute error will be less than the given value.

13. Exercise 1: $5 \times 10^{-4}$  
14. Exercise 2: $5 \times 10^{-4}$

15. Exercise 3: $10^{-3}$  
16. Exercise 4: $10^{-3}$

17. Exercise 5: $10^{-4}$  
18. Exercise 6: $10^{-4}$

19–22 True–False Determine whether the statement is true or false. Explain your answer.

19. The midpoint approximation, $M_n$, is the average of the left and right endpoint approximations, $L_n$ and $R_n$, respectively.

20. If $f(x)$ is concave down on the interval $(a, b)$, then the trapezoidal approximation $T_n$ underestimates $\int_a^b f(x) \, dx$.

21. The Simpson’s rule approximation $S_{2n}$ for $\int_a^b f(x) \, dx$ is a weighted average of the approximations $M_n$ and $T_n$, where $M_n$ is given twice the weight of $T_n$ in the average.

22. Simpson’s rule approximation $S_{2n}$ for $\int_a^b f(x) \, dx$ corresponds to $\int_a^b q(x) \, dx$, where the graph of $q$ is composed of 25 parabolic segments joined at points on the graph of $f$. 

23–24 Find a function $g(x)$ of the form

$g(x) = Ax^2 + Bx + C$

whose graph contains the points $(m - \Delta x, f(m - \Delta x)), (m, f(m)), (m + \Delta x, f(m + \Delta x))$, for the given function $f(x)$ and the given values of $m$ and $\Delta x$. Then verify Formula (11):

$\int_{m-\Delta x}^{m+\Delta x} g(x) \, dx = \frac{\Delta x}{3} [Y_0 + 4Y_1 + Y_2]$

where $Y_0 = f(m - \Delta x), Y_1 = f(m)$, and $Y_2 = f(m + \Delta x)$.

23. $f(x) = \frac{1}{x}; m = 3, \Delta x = 1$

24. $f(x) = \sin^2(\pi x); m = \frac{1}{2}, \Delta x = \frac{1}{2}$

25–30 Approximate the integral using Simpson’s rule $S_{10}$ and compare your answer to that produced by a calculating utility with a numerical integration capability. Express your answers to at least four decimal places.

25. $\int_{-1}^1 e^{-x^2} \, dx$

26. $\int_0^3 \frac{x}{\sqrt{2x^3 + 1}} \, dx$

27. $\int_0^2 x\sqrt{2 + x^3} \, dx$

28. $\int_0^\pi \frac{x}{2 + \sin x} \, dx$

29. $\int_0^1 \cos(x^2) \, dx$

30. $\int_1^3 (\ln x)^{3/2} \, dx$

31–32 The exact value of the given integral is $\pi$ (verify). Approximate the integral using (a) the midpoint approximation $M_{10}$, (b) the trapezoidal approximation $T_{10}$, and (c) Simpson’s rule approximation $S_{20}$ using Formula (7). Approximate the absolute error and express your answers to at least four decimal places.

31. $\int_0^2 \frac{8}{x^2 + 4} \, dx$

32. $\int_0^2 \frac{4}{\sqrt{9 - x^2}} \, dx$

33. In Example 8 we showed that taking $n = 14$ subdivisions ensures that the approximation of $\ln 2 = \int_1^2 \frac{1}{x} \, dx$ by Simpson’s rule is accurate to five decimal places. Confirm this by comparing the approximation of $\ln 2$ produced by Simpson’s rule with $n = 14$ to the value produced directly by your calculating utility.
34. In each part, determine whether a trapezoidal approximation would be an underestimate or an overestimate for the definite integral.
   
   (a) $\int_0^1 \cos(x^2) \, dx$  
   (b) $\int_0^{3/2} 2 \cos(x^2) \, dx$

35–36 Find a value of $n$ to ensure that the absolute error in approximating the integral by the midpoint approximation will be less than $10^{-4}$. Estimate the absolute error, and express your answers to at least four decimal places.

35. $\int_0^1 x \sin x \, dx$  
36. $\int_0^1 e^{\cos x} \, dx$

37–38 Show that the inequalities (12) and (13) are of no value in finding an upper bound on the absolute error that results from approximating the integral using either the midpoint approximation or the trapezoidal approximation.

37. $\int_0^1 x \sqrt{x} \, dx$  
38. $\int_0^1 \sin \sqrt{x} \, dx$

39–40 Use Simpson’s rule approximation $S_{10}$ to approximate the length of the curve over the stated interval. Express your answers to at least four decimal places.

39. $y = \sin x$ from $x = 0$ to $x = \pi$  
40. $y = x^{-2}$ from $x = 1$ to $x = 2$

**FOCUS ON CONCEPTS**

41. A graph of the speed $v$ versus time $t$ curve for a test run of a BMW 335i is shown in the accompanying figure. Estimate the speeds at times $t = 0, 5, 10, 15, 20, 25, 30$ s from the graph, convert to ft/s using $1 \text{mi/h} = \frac{22}{15} \text{ft/s}$, and use these speeds and Simpson’s rule to approximate the number of feet traveled during the first 30 s. Round your answer to the nearest foot. [Hint: Distance traveled $= \int_0^{30} v(t) \, dt$.]

**Source:** Data from Car and Driver Magazine, September 2010.

42. A graph of the acceleration $a$ versus time $t$ for an object moving on a straight line is shown in the accompanying figure. Estimate the accelerations at $t = 0, 1, 2, \ldots, 8$ seconds (s) from the graph and use Simpson’s rule to approximate the change in velocity from $t = 0$ to $t = 8$ s. Round your answer to the nearest tenth cm/s. [Hint: Change in velocity $= \int_0^8 a(t) \, dt$.]

43–46 Numerical integration methods can be used in problems where only measured or experimentally determined values of the integrand are available. Use Simpson’s rule to estimate the value of the relevant integral in these exercises.

43. The accompanying table gives the speeds, in miles per second, at various times for a test rocket that was fired upward from the surface of the Earth. Use these values to approximate the number of miles traveled during the first 180 s. Round your answer to the nearest tenth of a mile. [Hint: Distance traveled $= \int_0^{180} v(t) \, dt$.]

<table>
<thead>
<tr>
<th>TIME $t$ (s)</th>
<th>SPEED $v$ (mi/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
</tr>
<tr>
<td>30</td>
<td>0.03</td>
</tr>
<tr>
<td>60</td>
<td>0.08</td>
</tr>
<tr>
<td>90</td>
<td>0.16</td>
</tr>
<tr>
<td>120</td>
<td>0.27</td>
</tr>
<tr>
<td>150</td>
<td>0.42</td>
</tr>
<tr>
<td>180</td>
<td>0.65</td>
</tr>
</tbody>
</table>

**Table Ex-43**

44. The accompanying table gives the speeds of a bullet at various distances from the muzzle of a rifle. Use these values to approximate the number of seconds for the bullet to travel 1800 ft. Express your answer to the nearest hundredth of a second. [Hint: If $v$ is the speed of the bullet and $x$ is the distance traveled, then $v = dx/dt$ so that $dt/dx = 1/v$ and $t = \int_0^{1800} (1/v) \, dx$.]

<table>
<thead>
<tr>
<th>DISTANCE $x$ (ft)</th>
<th>SPEED $v$ (ft/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3100</td>
</tr>
<tr>
<td>300</td>
<td>2908</td>
</tr>
<tr>
<td>600</td>
<td>2725</td>
</tr>
<tr>
<td>900</td>
<td>2549</td>
</tr>
<tr>
<td>1200</td>
<td>2379</td>
</tr>
<tr>
<td>1500</td>
<td>2216</td>
</tr>
<tr>
<td>1800</td>
<td>2059</td>
</tr>
</tbody>
</table>

**Table Ex-44**

45. Measurements of a pottery shard recovered from an archaeological dig reveal that the shard came from a pot with a flat bottom and circular cross sections (see the accompanying
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Engineers want to construct a straight and level road 600 ft long and 75 ft wide by making a vertical cut through an intervening hill (see the accompanying figure). Heights of the hill above the centerline of the proposed road, as obtained at various points from a contour map of the region, are shown in the accompanying figure. To estimate the construction costs, the engineers need to know the volume of earth that must be removed. Approximate this volume, rounded to the nearest cubic foot. [Hint: First set up an integral for the cross-sectional area of the cut along the centerline of the road, then assume that the height of the hill does not vary between the centerline and edges of the road.]

<table>
<thead>
<tr>
<th>HORIZONTAL DISTANCE X (ft)</th>
<th>HEIGHT h (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>7</td>
</tr>
<tr>
<td>200</td>
<td>16</td>
</tr>
<tr>
<td>300</td>
<td>24</td>
</tr>
<tr>
<td>400</td>
<td>25</td>
</tr>
<tr>
<td>500</td>
<td>16</td>
</tr>
<tr>
<td>600</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure Ex-46

46. Let \( f(x) = \cos(x^2) \). (a) Use a CAS to approximate the maximum value of \( |f''(x)| \) on the interval \([0, 1]\). (b) How large must \( n \) be in the midpoint approximation of \( \int_0^1 f(x) \, dx \) to ensure that the absolute error is less than \( 5 \times 10^{-4} \)? Compare your result with that obtained in Example 9. (c) Estimate the integral using the midpoint approximation with the value of \( n \) obtained in part (b).

47. Let \( f(x) = \sqrt{1 + x^2} \). (a) Use a CAS to approximate the maximum value of \( |f''(x)| \) on the interval \([0, 1]\). (b) How large must \( n \) be in the midpoint approximation of \( \int_0^1 f(x) \, dx \) to ensure that the absolute error is less than \( 5 \times 10^{-4} \)? (c) Estimate the integral using the midpoint approximation with the value of \( n \) obtained in part (b).

48. Let \( f(x) = \sqrt{2 + x^2} \). (a) Use a CAS to approximate the maximum value of \( |f''(x)| \) on the interval \([0, 1]\). (b) How large must \( n \) be in the trapezoidal approximation of \( \int_0^1 f(x) \, dx \) to ensure that the absolute error is less than \( 10^{-3} \)? (c) Estimate the integral using the trapezoidal approximation with the value of \( n \) obtained in part (b).

49. Let \( f(x) = \cos(x - x^2) \). (a) Use a CAS to approximate the maximum value of \( |f^{(4)}(x)| \) on the interval \([0, 1]\). (b) How large must the value of \( n \) be in the approximation \( S_n \) of \( \int_0^1 f(x) \, dx \) by Simpson’s rule to ensure that the absolute error is less than \( 10^{-3} \)? (c) Estimate the integral using Simpson’s rule approximation \( S_n \) with the value of \( n \) obtained in part (b).

50. Let \( f(x) = \sqrt{2 + x^2} \). (a) Use a CAS to approximate the maximum value of \( |f^{(4)}(x)| \) on the interval \([0, 1]\). (b) How large must the value of \( n \) be in the approximation \( S_n \) of \( \int_0^1 f(x) \, dx \) by Simpson’s rule to ensure that the absolute error is less than \( 10^{-3} \)? (c) Estimate the integral using Simpson’s rule approximation \( S_n \) with the value of \( n \) obtained in part (b).

FOCUS ON CONCEPTS

51. (a) Verify that the average of the left and right endpoint approximations as given in Table 7.7.1 gives Formula (2) for the trapezoidal approximation. (b) Suppose that \( f \) is a continuous nonnegative function on the interval \([a, b]\) and partition \([a, b]\) with equally spaced points, \( a = x_0 < x_1 < \cdots < x_n = b \). Find the area of the trapezoid under the line segment joining points \((x_k, f(x_k))\) and \((x_{k+1}, f(x_{k+1}))\) and above the interval \([x_k, x_{k+1}]\). Show that the right side of Formula (2) is the sum of these trapezoidal areas (Figure 7.7.1).

52. Let \( f \) be a function that is positive, continuous, decreasing, and concave down on the interval \([a, b]\). Assuming that \([a, b]\) is subdivided into \( n \) equal subintervals, arrange the following approximations of \( \int_a^b f(x) \, dx \) in order of increasing value: left endpoint, right endpoint, midpoint, and trapezoidal.

53. Suppose that \( \Delta x > 0 \) and \( g(x) = Ax^2 + Bx + C \). Let \( m \) be a number and set \( Y_0 = g(m - \Delta x), Y_1 = g(m), \) and \( Y_2 = g(m + \Delta x) \). Verify Formula (11):

\[
\int_{m - \Delta x}^{m + \Delta x} g(x) \, dx = \frac{\Delta x}{3} [Y_0 + 4Y_1 + Y_2]
\]

54. Suppose that \( f \) is a continuous nonnegative function on the interval \([a, b]\), \( n \) is even, and \([a, b]\) is partitioned using \( n + 1 \) equally spaced points, \( a = x_0 < x_1 < \cdots < x_n = b \). Set \( y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n) \). Let \( g_1, g_2, \ldots, g_{n/2} \) be the quadratic functions of the form \( g_k(x) = Ax^2 + Bx + C \) so that (cont.)
the graph of $g_1$ passes through the points $(x_0, y_0)$, $(x_1, y_1)$, and $(x_2, y_2)$;

the graph of $g_2$ passes through the points $(x_2, y_2)$, $(x_3, y_3)$, and $(x_4, y_4)$;

... 

the graph of $g_{n/2}$ passes through the points $(x_{n-2}, y_{n-2})$, $(x_{n-1}, y_{n-1})$, and $(x_n, y_n)$.

Verify that Formula (8) computes the area under a piecewise quadratic function by showing that

$$
\sum_{j=1}^{n/2} \left( \int_{x_{j-1}}^{x_j} g_j(x) \, dx \right)
= \frac{1}{3} \left( \frac{b-a}{n} \right) \left[ y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n \right]
$$

7.8 Improper Integrals  

Up to now we have focused on definite integrals with continuous integrands and finite intervals of integration. In this section we will extend the concept of a definite integral to include infinite intervals of integration and integrands that become infinite within the interval of integration.

**Improper Integrals**

It is assumed in the definition of the definite integral

$$
\int_a^b f(x) \, dx
$$

that $[a, b]$ is a finite interval and that the limit that defines the integral exists; that is, the function $f$ is integrable. We observed in Theorems 5.5.2 and 5.5.8 that continuous functions are integrable, as are bounded functions with finitely many points of discontinuity. We also observed in Theorem 5.5.8 that functions that are not bounded on the interval of integration are not integrable. Thus, for example, a function with a vertical asymptote within the interval of integration would not be integrable.

Our main objective in this section is to extend the concept of a definite integral to allow for infinite intervals of integration and integrands with vertical asymptotes within the interval of integration. We will call the vertical asymptotes **infinite discontinuities**, and we will call
integrals with infinite intervals of integration or infinite discontinuities within the interval of integration improper integrals. Here are some examples:

- Improper integrals with infinite intervals of integration or infinite discontinuities within the interval of integration:

\[ \int_1^{\infty} \frac{dx}{x^2}, \int_0^0 e^x \, dx, \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} \]

- Improper integrals with infinite discontinuities in the interval of integration:

\[ \int_{-3}^{3} \frac{dx}{x^2}, \int_{1}^{2} \frac{dx}{x - 1}, \int_{0}^{\pi} \tan x \, dx \]

- Improper integrals with infinite discontinuities and infinite intervals of integration:

\[ \int_0^{\infty} \frac{dx}{\sqrt{x}}, \int_{-\infty}^{\infty} \frac{dx}{x^2 - 9}, \int_{1}^{\infty} \sec x \, dx \]

### INTEGRALS OVER INFINITE INTERVALS

To motivate a reasonable definition for improper integrals of the form

\[ \int_a^{\infty} f(x) \, dx \]

let us begin with the case where \( f \) is continuous and nonnegative on \([a, +\infty)\), so we can think of the integral as the area under the curve \( y = f(x) \) over the interval \([a, +\infty)\) (Figure 7.8.1). At first, you might be inclined to argue that this area is infinite because the region has infinite extent. However, such an argument would be based on vague intuition rather than precise mathematical logic, since the concept of area has only been defined over intervals of finite extent. Thus, before we can make any reasonable statements about the area of the region in Figure 7.8.1, we need to begin by defining what we mean by the area of this region. For that purpose, it will help to focus on a specific example.

Suppose we are interested in the area \( A \) of the region that lies below the curve \( y = \frac{1}{x^2} \) and above the interval \([1, +\infty)\) on the \( x \)-axis. Instead of trying to find the entire area at once, let us begin by calculating the portion of the area that lies above a finite interval \([1, b]\), where \( b > 1 \) is arbitrary. That area is

\[ \int_1^{b} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_1^{b} = 1 - \frac{1}{b} \]  

(Figure 7.8.2). If we now allow \( b \) to increase so that \( b \to +\infty \), then the portion of the area over the interval \([1, b]\) will begin to fill out the area over the entire interval \([1, +\infty)\) (Figure 7.8.3), and hence we can reasonably define the area \( A \) under \( y = \frac{1}{x^2} \) over the interval \([1, +\infty)\) to be

\[ A = \int_1^{\infty} \frac{dx}{x^2} = \lim_{b \to +\infty} \int_1^{b} \frac{dx}{x^2} = \lim_{b \to +\infty} \left( 1 - \frac{1}{b} \right) = 1 \]  

Thus, the area has a finite value of 1 and is not infinite as we first conjectured.

With the preceding discussion as our guide, we make the following definition (which is applicable to functions with both positive and negative values). **Figure 7.8.1**

\[ y = \frac{1}{x^2}, \text{Area} = \int_1^{\infty} \frac{dx}{x^2} = 1 - \frac{1}{b} \]  

**Figure 7.8.2**

\[ y = \frac{1}{x^2}, \text{Area} = \int_1^{b} \frac{dx}{x^2} = 1 - \frac{1}{b} \]  

**Figure 7.8.3**

\[ y = \frac{1}{x^2}, \text{Area} = \int_1^{\infty} \frac{dx}{x^2} = \lim_{b \to +\infty} \left( 1 - \frac{1}{b} \right) = 1 \]
If $f$ is nonnegative over the interval $[a, +\infty)$, then the improper integral in Definition 7.8.1 can be interpreted to be the area under the graph of $f$ over the interval $[a, +\infty)$. If the integral converges, then the area is finite and equal to the value of the integral, and if the integral diverges, then the area is regarded to be infinite.

7.8 Improper Integrals

7.8.1 Definition

The improper integral of $f$ over the interval $[a, +\infty)$ is defined to be

$$\int_a^{+\infty} f(x) \, dx = \lim_{b \to +\infty} \int_a^b f(x) \, dx$$

In the case where the limit exists, the improper integral is said to converge, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to diverge, and it is not assigned a value.

Example 1

Evaluate

(a) $\int_1^{+\infty} \frac{dx}{x^3}$  (b) $\int_1^{+\infty} \frac{dx}{x}$

Solution (a).

Following the definition, we replace the infinite upper limit by a finite upper limit $b$, and then take the limit of the resulting integral. This yields

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^3} = \lim_{b \to +\infty} \left[ -\frac{1}{2x^2} \right]_1^b = \lim_{b \to +\infty} \left( \frac{1}{2} - \frac{1}{2b^2} \right) = \frac{1}{2}$$

Since the limit is finite, the integral converges and its value is $1/2$.

Solution (b).

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b \to +\infty} \left[ \ln x \right]_1^b = \lim_{b \to +\infty} \ln b = +\infty$$

In this case the integral diverges and hence has no value.

Figure 7.8.4

Because the functions $1/x^3, 1/x^2$, and $1/x$ are nonnegative over the interval $[1, +\infty)$, it follows from (1) and the last example that over this interval the area under $y = 1/x^3$ is $1/2$, the area under $y = 1/x^2$ is 1, and the area under $y = 1/x$ is infinite. However, on the surface the graphs of the three functions seem very much alike (Figure 7.8.4), and there is nothing to suggest why one of the areas should be infinite and the other two finite. One explanation is that $1/x^3$ and $1/x^2$ approach zero more rapidly than $1/x$ as $x \to +\infty$, so that the area over the interval $[1, b]$ accumulates less rapidly under the curves $y = 1/x^3$ and $y = 1/x^2$ than under $y = 1/x$ as $b \to +\infty$, and the difference is just enough that the first two areas are finite and the third is infinite.

Example 2

For what values of $p$ does the integral $\int_1^{+\infty} \frac{dx}{x^p}$ converge?

Solution.

We know from the preceding example that the integral diverges if $p = 1$, so let us assume that $p \neq 1$. In this case we have

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b \to +\infty} \int_1^b x^{-p} \, dx = \lim_{b \to +\infty} \left[ \frac{x^{1-p}}{1-p} \right]_1^b = \lim_{b \to +\infty} \left[ \frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right]$$

If $p > 1$, then the exponent $1-p$ is negative and $b^{1-p} \to 0$ as $b \to +\infty$; and if $p < 1$, then the exponent $1-p$ is positive and $b^{1-p} \to +\infty$ as $b \to +\infty$. Thus, the integral converges if $p > 1$ and diverges otherwise. In the convergent case the value of the integral is

$$\int_1^{+\infty} \frac{dx}{x^p} = \left[ 0 - \frac{1}{1-p} \right] = \frac{1}{p-1} \quad (p > 1) \quad \blacksquare$$
The following theorem summarizes this result.

**7.8.2 Theorem**

\[
\int_{1}^{+\infty} \frac{dx}{x^p} = \begin{cases} 
\frac{1}{p-1} & \text{if } p > 1 \\
\text{diverges} & \text{if } p \leq 1
\end{cases}
\]

**Example 3** Evaluate \( \int_{0}^{+\infty} (1-x)e^{-x} \, dx \).

**Solution.** We begin by evaluating the indefinite integral using integration by parts. Setting \( u = 1-x \) and \( dv = e^{-x} \, dx \) yields

\[
\int (1-x)e^{-x} \, dx = -e^{-x}(1-x) - \int e^{-x} \, dx = -e^{-x} + xe^{-x} + C = xe^{-x} + C
\]

Thus,

\[
\int_{0}^{+\infty} (1-x)e^{-x} \, dx = \lim_{b \to +\infty} \int_{0}^{b} (1-x)e^{-x} \, dx = \lim_{b \to +\infty} \left[ xe^{-x} \right]_{0}^{b} = \lim_{b \to +\infty} \frac{b}{e^b}
\]

The limit is an indeterminate form of type \( \infty/\infty \), so we will apply L'Hôpital's rule by differentiating the numerator and denominator with respect to \( b \). This yields

\[
\int_{0}^{+\infty} (1-x)e^{-x} \, dx = \lim_{b \to +\infty} \frac{1}{e^b} = 0
\]

We can interpret this to mean that the net signed area between the graph of \( y = (1-x)e^{-x} \) and the interval \([0, +\infty)\) is 0 (Figure 7.8.5).

**7.8.3 Definition** The improper integral of \( f \) over the interval \( (-\infty, b] \) is defined to be

\[
\int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx
\]

(2)

The integral is said to converge if the limit exists and diverge if it does not.

The improper integral of \( f \) over the interval \( (-\infty, +\infty) \) is defined as

\[
\int_{-\infty}^{+\infty} f(x) \, dx = \int_{-\infty}^{c} f(x) \, dx + \int_{c}^{+\infty} f(x) \, dx
\]

(3)

where \( c \) is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.

**Example 4** Evaluate \( \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} \).

**Solution.** We will evaluate the integral by choosing \( c = 0 \) in (3). With this value for \( c \) we obtain

\[
\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{b \to +\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b \to +\infty} \left[ \tan^{-1} x \right]_{0}^{b} = \lim_{b \to +\infty} (\tan^{-1} b) = \frac{\pi}{2}
\]

\[
\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \left[ \tan^{-1} x \right]_{a}^{0} = \lim_{a \to -\infty} (-\tan^{-1} a) = \frac{\pi}{2}
\]

Although we usually choose \( c = 0 \) in (3), the choice does not matter because it can be proved that neither the convergence nor the value of the integral is affected by the choice of \( c \).
Thus, the integral converges and its value is

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since the integrand is nonnegative on the interval $$(-\infty, +\infty)$$, the integral represents the area of the region shown in Figure 7.8.6.

## 7.8 Improper Integrals

### Integrals Whose Integrands Have Infinite Discontinuities

Next we will consider improper integrals whose integrands have infinite discontinuities. We will start with the case where the interval of integration is a finite interval $$[a, b]$$ and the infinite discontinuity occurs at the right-hand endpoint.

To motivate an appropriate definition for such an integral let us consider the case where $$f$$ is nonnegative on $$[a, b]$$, so we can interpret the improper integral

$$\int_{a}^{b} f(x) \, dx$$

as the area of the region in Figure 7.8.7. The problem of finding the area of this region is complicated by the fact that it extends indefinitely in the positive $$y$$-direction. However, instead of trying to find the entire area at once, we can proceed indirectly by calculating the portion of the area over the interval $$[a, k]$$, where $$a \leq k < b$$, and then letting $$k$$ approach $$b$$ to fill out the area of the entire region (Figure 7.8.7). Motivated by this idea, we make the following definition.

**7.8.4 Definition** If $$f$$ is continuous on the interval $$[a, b]$$, except for an infinite discontinuity at $$b$$, then the improper integral of $$f$$ over the interval $$[a, b]$$ is defined as

$$\int_{a}^{b} f(x) \, dx = \lim_{k \to b^{-}} \int_{a}^{k} f(x) \, dx$$

In the case where the indicated limit exists, the improper integral is said to converge, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to diverge, and it is not assigned a value.

**Example 5** Evaluate $$\int_{0}^{1} \frac{dx}{\sqrt{1-x}}$$.

**Solution.** The integral is improper because the integrand approaches $$+\infty$$ as $$x$$ approaches the upper limit 1 from the left (Figure 7.8.8). From (4),

$$\int_{0}^{1} \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^{-}} \int_{0}^{k} \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^{-}} \left[ -2\sqrt{1-k} + 2 \right]_{0}^{k}$$

$$= \lim_{k \to 1^{-}} \left[ -2\sqrt{1-k} + 2 \right] = 2$$

Improper integrals with an infinite discontinuity at the left-hand endpoint or inside the interval of integration are defined as follows.
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7.8.5 Definition If \( f \) is continuous on the interval \([a, b]\), except for an infinite discontinuity at \( a \), then the improper integral of \( f \) over the interval \([a, b]\) is defined as

\[
\int_a^b f(x) \, dx = \lim_{k \to a^+} \int_k^b f(x) \, dx
\]

The integral is said to converge if the indicated limit exists and diverge if it does not.

If \( f \) is continuous on the interval \([a, b]\), except for an infinite discontinuity at a point \( c \) in \((a, b)\), then the improper integral of \( f \) over the interval \([a, b]\) is defined as

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

where the two integrals on the right side are themselves improper. The improper integral on the left side is said to converge if both terms on the right side converge and diverge if either term on the right side diverges (Figure 7.8.9).

Example 6 Evaluate

(a) \[ \int_1^2 \frac{dx}{1 - x} \]  
(b) \[ \int_1^4 \frac{dx}{(x - 2)^{2/3}} \]

Solution (a). The integral is improper because the integrand approaches \(-\infty\) as \( x \) approaches the lower limit \( 1 \) from the right (Figure 7.8.10). From Definition 7.8.5 we obtain

\[
\int_1^2 \frac{dx}{1 - x} = \lim_{k \to 1^+} \int_k^2 \frac{dx}{1 - x} = \lim_{k \to 1^+} \left[ -\ln |1 - x| \right]_k^2 = \lim_{k \to 1^+} \left[ -\ln |1 - 1| + \ln |1 - k| \right] = \lim_{k \to 1^+} \ln |1 - k| = -\infty
\]

so the integral diverges.

Solution (b). The integral is improper because the integrand approaches \(+\infty\) at \( x = 2 \), which is inside the interval of integration. From Definition 7.8.5 we obtain

\[
\int_1^4 \frac{dx}{(x - 2)^{2/3}} = \int_1^2 \frac{dx}{(x - 2)^{2/3}} + \int_2^4 \frac{dx}{(x - 2)^{2/3}}
\]

and we must investigate the convergence of both improper integrals on the right. Since

\[
\int_1^2 \frac{dx}{(x - 2)^{2/3}} = \lim_{k \to 2^-} \int_k^2 \frac{dx}{(x - 2)^{2/3}} = \lim_{k \to 2^-} \left[ 3(k - 2)^{1/3} - 3(1 - 2)^{1/3} \right] = 3
\]

\[
\int_2^4 \frac{dx}{(x - 2)^{2/3}} = \lim_{k \to 2^+} \int_k^4 \frac{dx}{(x - 2)^{2/3}} = \lim_{k \to 2^+} \left[ 3(4 - 2)^{1/3} - 3(k - 2)^{1/3} \right] = 3 \sqrt[3]{2}
\]

we have from (7) that

\[
\int_1^4 \frac{dx}{(x - 2)^{2/3}} = 3 + 3 \sqrt[3]{2}
\]
It is sometimes tempting to apply the Fundamental Theorem of Calculus directly to an improper integral without taking the appropriate limits. To illustrate what can go wrong with this procedure, suppose we fail to recognize that the integral
\[
\int_0^2 \frac{dx}{(x-1)^2}
\]
is improper and mistakenly evaluate this integral as
\[
-\frac{1}{x-1} \bigg|_0^2 = -1 - (1) = -2
\]
This result is clearly incorrect because the integrand is never negative and hence the integral cannot be negative! To evaluate (8) correctly we should first write
\[
\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}
\]
and then treat each term as an improper integral. For the first term,
\[
\int_0^1 \frac{dx}{(x-1)^2} = \lim_{k \to 1^-} \int_k^1 \frac{dx}{(x-1)^2} = \lim_{k \to 1^-} \left[ -\frac{1}{x-1} \right]_k^1 = +\infty
\]
so (8) diverges.

**Example 7** Derive the formula for the circumference of a circle of radius \( r \).

**Solution.** For convenience, let us assume that the circle is centered at the origin, in which case its equation is \( x^2 + y^2 = r^2 \). We will find the arc length of the portion of the circle that lies in the first quadrant and then multiply by 4 to obtain the total circumference (Figure 7.8.11).

Since the equation of the upper semicircle is \( y = \sqrt{r^2 - x^2} \), it follows from Formula (4) of Section 6.4 that the circumference \( C \) is
\[
C = 4 \int_0^r \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = 4 \int_0^r \sqrt{1 + \frac{x}{\sqrt{r^2 - x^2}}} \, dx
\]
\[
= 4r \int_0^r \frac{dx}{\sqrt{r^2 - x^2}}
\]
This integral is improper because of the infinite discontinuity at \( x = r \), and hence we evaluate it by writing
\[
C = 4r \lim_{k \to r^-} \int_0^k \frac{dx}{\sqrt{r^2 - x^2}}
\]
\[
= 4r \lim_{k \to r^-} \left[ \sin^{-1} \left( \frac{x}{r} \right) \right]_0^k
\]
\[
= 4r \lim_{k \to r^-} \left[ \sin^{-1} \left( \frac{k}{r} \right) - \sin^{-1} 0 \right]
\]
\[
= 4r \left[ \sin^{-1} 1 - \sin^{-1} 0 \right] = 4r \left[ \frac{\pi}{2} - 0 \right] = 2\pi r
\]
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Quick Check Exercises 7.8  (See page 557 for answers.)

1. In each part, determine whether the integral is improper, and if so, explain why. Do not evaluate the integrals.
   (a) \( \int_{\pi/4}^{3\pi/4} \cot x \, dx \)
   (b) \( \int_{\pi/4}^{\pi} \cot x \, dx \)
   (c) \( \int_{0}^{\infty} \frac{1}{x^2 + 1} \, dx \)
   (d) \( \int_{1}^{\infty} \frac{1}{x^2 - 1} \, dx \)

2. Express each improper integral in Quick Check Exercise 1 in terms of one or more appropriate limits. Do not evaluate the limits.

Exercises Set 7.8  (See pages 559 for graphs.)

1. In each part, determine whether the integral is improper, and if so, explain why.
   (a) \( \int_{3}^{5} \frac{dx}{x - 3} \)
   (b) \( \int_{1}^{5} \frac{dx}{x + 3} \)
   (c) \( \int_{1}^{\infty} \ln x \, dx \)
   (d) \( \int_{1}^{\infty} e^{-x} \, dx \)
   (e) \( \int_{0}^{\infty} \frac{dx}{\sqrt{x - 1}} \)
   (f) \( \int_{0}^{\pi/4} \tan x \, dx \)

2. In each part, determine all values of \( p \) for which the integral is improper.
   (a) \( \int_{0}^{1} \frac{dx}{x^p} \)
   (b) \( \int_{1}^{2} \frac{dx}{x - p} \)
   (c) \( \int_{1}^{\infty} e^{-px} \, dx \)

3–32 Evaluate the integrals that converge.

3. \( \int_{0}^{\infty} e^{-2x} \, dx \)
4. \( \int_{0}^{1} \frac{x}{1 + x^2} \, dx \)
5. \( \int_{3}^{\infty} \frac{dx}{x^2 - 1} \)
6. \( \int_{0}^{1} e^{-x^2} \, dx \)
7. \( \int_{e}^{\infty} \frac{1}{x \ln x} \, dx \)
8. \( \int_{2}^{\infty} \frac{1}{x \sqrt{x}} \, dx \)
9. \( \int_{0}^{\infty} \frac{e^x}{2x - 1} \, dx \)
10. \( \int_{0}^{\infty} e^{3x} \, dx \)
11. \( \int_{0}^{\infty} e^{ax} \, dx \)
12. \( \int_{0}^{\infty} \frac{x}{3 - 2e^x} \, dx \)
13. \( \int_{0}^{\infty} x \, dx \)
14. \( \int_{0}^{\infty} \frac{x}{\sqrt{x^2 + 2}} \, dx \)
15. \( \int_{0}^{\infty} \frac{dx}{x^2 + 3} \)
16. \( \int_{0}^{\infty} e^{-x^2} \, dx \)
17. \( \int_{0}^{4} \frac{dx}{x - 4} \)
18. \( \int_{0}^{\infty} \frac{dx}{\sqrt{x}} \)
19. \( \int_{0}^{\infty} \frac{dx}{\tan x} \)
20. \( \int_{0}^{\infty} \frac{dx}{\sqrt{4 - x}} \)
21. \( \int_{0}^{\infty} \frac{dx}{\sqrt{1 - x^2}} \)
22. \( \int_{0}^{\infty} \frac{dx}{\sqrt{9 - x^2}} \)

33–36 True–False  Determine whether the statement is true or false. Explain your answer.

33. \( \int_{1}^{\infty} x^{-4/3} \, dx \) converges to 3.
34. If \( f \) is continuous on \( [a, +\infty) \) and \( \lim_{x \to +\infty} f(x) = 1 \), then \( \int_{a}^{\infty} f(x) \, dx \) converges.
35. \( \int_{1}^{\infty} \frac{1}{x(x - 3)} \, dx \) is an improper integral.
36. \( \int_{1}^{\infty} \frac{1}{x^3} \, dx = 0 \)

37–40 Make the \( u \)-substitution and evaluate the resulting definite integral.

37. \( \int_{0}^{\infty} e^{-\sqrt{x}} \, dx; \ u = \sqrt{x} \) [Note: \( u \to +\infty \) as \( x \to +\infty \).]
38. \( \int_{0}^{1/2} \frac{dx}{\sqrt{x + 4}}; \ u = \sqrt{x} \) [Note: \( u \to +\infty \) as \( x \to +\infty \).]
39. \( \int_{0}^{\infty} \frac{e^{-x}}{\sqrt{1 - e^{-x}}} \, dx; \ u = 1 - e^{-x} \) [Note: \( u \to 1 \) as \( x \to +\infty \).]
40. \( \int_0^\infty \frac{e^{-x}}{\sqrt{1 - e^{-2x}}} \, dx \); \( u = e^{-x} \)

41–42 Express the improper integral as a limit, and then evaluate that limit with a CAS. Confirm the answer by evaluating the integral directly with the CAS. ■

43. In each part, try to evaluate the integral exactly with a CAS. If your result is not a simple numerical answer, then use the CAS to find a numerical approximation of the integral.

(a) \( \int_{-\infty}^{\infty} \frac{1}{x^8 + x + 1} \, dx \) \hspace{1cm} (b) \( \int_0^\infty \frac{1}{\sqrt{1 + x^3}} \, dx \)

(c) \( \int_1^\infty \ln x \, dx \) \hspace{1cm} (d) \( \int_1^\infty \frac{\sin x}{x^2} \, dx \)

44. In each part, confirm the result with a CAS.

(a) \( \int_0^\infty \frac{\sin x}{\sqrt{x}} \, dx = \sqrt{\pi}/2 \) \hspace{1cm} (b) \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \)

(c) \( \int_0^\infty \ln x \frac{dx}{x} = -\pi^2/12 \)

45. Find the length of the curve \( y = (4 - x^{2/3})^{1/2} \) over the interval [0, 8].

46. Find the length of the curve \( y = \sqrt{4 - x^2} \) over the interval [0, 2].

47–48 Use L'Hôpital's rule to help evaluate the improper integral. ■

47. \( \int_0^1 \ln x \, dx \) \hspace{1cm} 48. \( \int_1^\infty \frac{\ln x}{x^2} \, dx \)

49. Find the area of the region between the x-axis and the curve \( y = e^{-3x} \) for \( x \geq 0 \).

50. Find the area of the region between the x-axis and the curve \( y = 8/(x^2 - 4) \) for \( x \geq 4 \).

51. Suppose that the region between the x-axis and the curve \( y = e^{-x} \) for \( x \geq 0 \) is revolved about the x-axis. (a) Find the volume of the solid that is generated. (b) Find the surface area of the solid.

**Focus on Concepts**

52. Suppose that \( f \) and \( g \) are continuous functions and that

\[
0 \leq f(x) \leq g(x)
\]

if \( x \geq a \). Give a reasonable informal argument using areas to explain why the following results are true.

(a) If \( \int_a^\infty f(x) \, dx \) diverges, then \( \int_a^\infty g(x) \, dx \) diverges.

(b) If \( \int_a^\infty g(x) \, dx \) converges, then \( \int_a^\infty f(x) \, dx \) converges and \( \int_a^\infty f(x) \, dx \leq \int_a^\infty g(x) \, dx \).

[Note: The results in this exercise are sometimes called **comparison tests** for improper integrals.]

53–56 Use the results in Exercise 52. ■

53. (a) Confirm graphically and algebraically that \( e^{-x^2} \leq e^{-x} \) (\( x \geq 1 \))

(b) Evaluate the integral \( \int_1^\infty e^{-x} \, dx \)

(c) What does the result obtained in part (b) tell you about the integral \( \int_1^\infty e^{-x^2} \, dx \)?

54. (a) Confirm graphically and algebraically that

\[
\frac{1}{2x + 1} \leq \frac{e^x}{2x + 1} \quad (x \geq 0)
\]

(b) Evaluate the integral \( \int_0^\infty \frac{dx}{2x + 1} \)

(c) What does the result obtained in part (b) tell you about the integral \( \int_0^\infty e^x/(2x + 1) \, dx \)?

55. Let \( R \) be the region to the right of \( x = 1 \) that is bounded by the x-axis and the curve \( y = 1/x \). When this region is revolved about the x-axis it generates a solid whose surface is known as **Gabriel's Horn** (for reasons that should be clear from the accompanying figure). Show that the solid has a finite volume but its surface has an infinite area. [Note: It has been suggested that if one could saturate the interior of the solid with paint and allow it to seep through to the surface, then one could paint an infinite surface with a finite amount of paint! What do you think?]

56. In each part, use Exercise 52 to determine whether the integral converges or diverges. If it converges, then use part (b) of that exercise to find an upper bound on the value of the integral.

(a) \( \int_2^\infty \frac{\sqrt{x^3 + 1}}{x} \, dx \) \hspace{1cm} (b) \( \int_2^\infty \frac{x}{x^3 + 1} \, dx \)

(c) \( \int_0^\infty \frac{xe^x}{2x + 1} \, dx \)

---

**Figure Ex-55**
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**FOCUS ON CONCEPTS**

57. Sketch the region whose area is
\[ \int_0^{\infty} \frac{dx}{1 + x^2} \]

and use your sketch to show that
\[ \int_0^{\infty} \frac{dx}{1 + x^2} = \int_0^1 \sqrt{1 - y} \, dy \]

58. (a) Give a reasonable informal argument, based on areas, that explains why the integrals
\[ \int_0^{\infty} \sin x \, dx \quad \text{and} \quad \int_0^{\infty} \cos x \, dx \]
diverge.
(b) Show that \( \int_0^{\infty} \cos \frac{x}{\sqrt{x}} \, dx \) diverges.

59. In electromagnetic theory, the magnetic potential at a point on the axis of a circular coil is given by
\[ u = \frac{2\pi NIr}{k} \int_0^{\infty} \frac{dx}{(r^2 + x^2)^{3/2}} \]
where \( N, I, r, k, \) and \( a \) are constants. Find \( u \).

60. The average speed, \( \bar{v} \), of the molecules of an ideal gas is given by
\[ \bar{v} = \frac{4}{\sqrt{\pi}} \left( \frac{M}{2RT} \right)^{3/2} \int_0^{\infty} v^3 e^{-Mv^2/(2RT)} \, dv \]
and the root-mean-square speed, \( v_{rms} \), by
\[ v_{rms} = \frac{4}{\sqrt{\pi}} \left( \frac{M}{2RT} \right)^{3/2} \int_0^{\infty} v^{1/2} e^{-Mv^2/(2RT)} \, dv \]
where \( v \) is the molecular speed, \( T \) is the gas temperature, \( M \) is the molecular weight of the gas, and \( R \) is the gas constant.

(a) Use a CAS to show that
\[ \int_0^{\infty} x^3 e^{-ax^2} \, dx = \frac{1}{2a^3}, \quad a > 0 \]
and use this result to show that \( \bar{v} = \sqrt{8RT/(\pi M)} \).
(b) Use a CAS to show that
\[ \int_0^{\infty} x^4 e^{-ax^2} \, dx = \frac{3\sqrt{\pi}}{8a^5}, \quad a > 0 \]
and use this result to show that \( v_{rms} = \sqrt{3RT/M} \).

61–62  Medication can be administered to a patient using a variety of methods. For a given method, let \( c(t) \) denote the concentration of medication in the patient’s bloodstream (measured in mg/L) \( t \) hours after the dose is given. The area under the curve \( c = c(t) \) over the time interval \([0, +\infty)\) indicates the “availability” of the medication for the patient’s body. Determine which method provides the greater availability.

61. Method 1: \( c_1(t) = 5(e^{-0.2t} - e^{-t}) \);
Method 2: \( c_2(t) = 4(e^{-0.2t} - e^{-3t}) \)

62. Method 1: \( c_1(t) = 6(e^{-0.4t} - e^{-1.3t}) \);
Method 2: \( c_2(t) = 5(e^{-0.4t} - e^{-3t}) \)

63. In Exercise 25 of Section 6.6, we determined the work required to lift a 6000 lb satellite to an orbital position that is 1000 mi above the Earth’s surface. The ideas discussed in that exercise will be needed here.
(a) Find a definite integral that represents the work required to lift a 6000 lb satellite to a position \( b \) miles above the Earth’s surface.
(b) Find a definite integral that represents the work required to lift a 6000 lb satellite an “infinite distance” above the Earth’s surface. Evaluate the integral. [Note: The result obtained here is sometimes called the work required to “escape” the Earth’s gravity.]

64–65  A transform is a formula that converts or “transforms” one function into another. Transforms are used in applications to convert a difficult problem into an easier problem whose solution can then be used to solve the original difficult problem. The Laplace transform of a function \( f(t) \), which plays an important role in the study of differential equations, is denoted by \( \mathcal{L}[f(t)] \) and is defined by
\[ \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) \, dt \]
In this formula \( s \) is treated as a constant in the integration process; thus, the Laplace transform has the effect of transforming \( f(t) \) into a function of \( s \). Use this formula in these exercises.

64. Show that
(a) \( \mathcal{L}[1] = \frac{1}{s}, \quad s > 0 \)
(b) \( \mathcal{L}[e^{at}] = \frac{1}{s - a}, \quad s > a \)
(c) \( \mathcal{L}[\sin t] = \frac{1}{s^2 + 1}, \quad s > 0 \)
(d) \( \mathcal{L}[\cos t] = \frac{s}{s^2 + 1}, \quad s > 0 \).

65. In each part, find the Laplace transform.
(a) \( f(t) = t^2, \quad s > 0 \)
(b) \( f(t) = t^2, \quad s > 0 \)
(c) \( f(t) = \begin{cases} 0, & t < 3 \\ 1, & t \geq 3 \end{cases}, \quad s > 0 \)

66. Later in the text, we will show that
\[ \int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \]
Confirm that this is reasonable by using a CAS or a calculator with a numerical integration capability.

67. Use the result in Exercise 66 to show that
(a) \( \int_0^{\infty} e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{a}, \quad a > 0 \)
(b) \( \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} \, dx = 1, \quad \sigma > 0 \).

68–69  A convergent improper integral over an infinite interval can be approximated by first replacing the infinite limit(s) of integration by finite limit(s), then using a numerical integration technique, such as Simpson’s rule, to approximate the integral with finite limit(s). This technique is illustrated in these exercises.
68. Suppose that the integral in Exercise 66 is approximated by first writing it as
\[ \int_{0}^{t^\pi} e^{-x^2} \, dx = \int_{0}^{K} e^{-x^2} \, dx + \int_{K}^{t^\pi} e^{-x^2} \, dx \]
then dropping the second term, and then applying Simpson’s rule to the integral
\[ \int_{0}^{K} e^{-x^2} \, dx \]

The resulting approximation has two sources of error: the error from Simpson’s rule and the error
\[ E = \int_{K}^{t^\pi} e^{-x^2} \, dx \]
that results from discarding the second term. We call \( E \) the truncation error.
(a) Approximate the integral in Exercise 66 by applying Simpson’s rule with \( n = 10 \) subdivisions to the integral
\[ \int_{0}^{3} e^{-x^2} \, dx \]
Round your answer to four decimal places and compare it to \( \frac{1}{\sqrt{\pi}} \) rounded to four decimal places.
(b) Use the result that you obtained in Exercise 52 and the fact that \( e^{-x^2} \leq e^{-x^3} \) for \( x \geq 3 \) to show that the truncation error for the approximation in part (a) satisfies
\[ 0 < E < 2 \times 10^{-4} \]

69. (a) It can be shown that
\[ \int_{0}^{+\pi} \frac{1}{x^6+1} \, dx = \frac{\pi}{3} \]
Approximate this integral by applying Simpson’s rule with \( n = 20 \) subdivisions to the integral
\[ \int_{0}^{4} \frac{1}{x^6+1} \, dx \]
Round your answer to three decimal places and compare it to \( \pi/3 \) rounded to three decimal places.

\[ \text{QUICK CHECK ANSWERS 7.8} \]

1. (a) improper, since \( \cot x \) has an infinite discontinuity at \( x = \pi \)
   (b) improper, since there is an infinite interval of integration
   (c) improper, since there is an infinite interval of integration and the integrand has an infinite discontinuity at \( x = 1 \)
   (d) improper, since there is an infinite interval of integration and the integrand has an infinite discontinuity at \( x = 1 \)

2. (b) \( \lim_{b \to a^-} \int_{a}^{b} \frac{1}{x^2+1} \, dx \)
   (c) \( \lim_{b \to a^+} \int_{a}^{b} \frac{1}{x^2+1} \, dx \)
   (d) \( \lim_{b \to \infty} \int_{a}^{b} \frac{1}{x^2+1} \, dx \)

3. \( \frac{1}{p-1}; p > 1 \)

4. (a) 1 (b) diverges (c) diverges (d) 3

\[ \text{CHAPTER 7 REVIEW EXERCISES} \]

1–6 Evaluate the given integral with the aid of an appropriate \( u \)-substitution.

1. \( \int \sqrt[4]{4+9x} \, dx \)
2. \( \int \frac{1}{\sec \pi x} \, dx \)
3. \( \int \sqrt{\cos x} \sin x \, dx \)
4. \( \int \frac{dx}{x \ln x} \)
5. \( \int x \tan^2 (x^2) \sec^2 (x^2) \, dx \)
6. \( \int \frac{\sqrt{x}}{x+9} \, dx \)

7. (a) Evaluate the integral
\[ \int \frac{1}{\sqrt{2x-x^2}} \, dx \]
three ways: using the substitution \( u = \sqrt{x} \), using the substitution \( u = \sqrt{2} - x \), and completing the square.
(b) Show that the answers in part (a) are equivalent.
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8. Evaluate the integral \( \int_0^1 \frac{x^3}{\sqrt{x^2 + 1}} \, dx \)
   (a) using integration by parts
   (b) using the substitution \( u = \sqrt{x^2 + 1} \).

9–12 Use integration by parts to evaluate the integral.

9. \( \int x e^{-x} \, dx \)
10. \( \int x \sin 2x \, dx \)
11. \( \int \ln(2x + 3) \, dx \)
12. \( \int \frac{1}{\sqrt{x^2 + 1}} \, dx \)

13. Evaluate \( \int 8x^4 \cos 2x \, dx \) using tabular integration by parts.

14. A particle moving along the \( x \)-axis has velocity function \( v(t) = t^2 e^{-t} \). How far does the particle travel from time \( t = 0 \) to \( t = 5 \)?

15–20 Evaluate the integral.

15. \( \int \sin^2 5\theta \, d\theta \)
16. \( \int \sin^3 2x \cos^2 2x \, dx \)
17. \( \int \sin x \cos 2x \, dx \)
18. \( \int_0^{\pi/6} \sin x \cos 4x \, dx \)
19. \( \int \sin^4 2x \, dx \)
20. \( \int x \cos^3 (x^2) \, dx \)

21–26 Evaluate the integral by making an appropriate trigonometric substitution.

21. \( \int \frac{x^2}{\sqrt{9 - x^2}} \, dx \)
22. \( \int \frac{dx}{x^2 \sqrt{16 - x^2}} \)
23. \( \int \frac{dx}{\sqrt{x^2 - 1}} \)
24. \( \int \frac{x^2}{\sqrt{x^2 - 25}} \, dx \)
25. \( \int \frac{x^2}{\sqrt{9 + x^2}} \, dx \)
26. \( \int \frac{1}{\sqrt{1 + 4x^2}} \, dx \)

27–32 Evaluate the integral using the method of partial fractions.

27. \( \int \frac{dx}{x^2 + 3x - 4} \)
28. \( \int \frac{dx}{x^2 + 8x + 7} \)
29. \( \int \frac{x^2 + 2}{x + 2} \, dx \)
30. \( \int \frac{x^2 + x - 16}{(x - 1)(x - 3)^2} \, dx \)
31. \( \int \frac{x^2}{(x + 2)^3} \, dx \)
32. \( \int \frac{dx}{x^3 + x} \)

33. Consider the integral \( \int \frac{1}{x^3 - x} \, dx \).
   (a) Evaluate the integral using the substitution \( x = \sec \theta \).
   For what values of \( x \) is your result valid?
   (b) Evaluate the integral using the substitution \( x = \sin \theta \).
   For what values of \( x \) is your result valid?
   (c) Evaluate the integral using the method of partial fractions.
   For what values of \( x \) is your result valid?

34. Find the area of the region that is enclosed by the curves \( y = (x - 3)/(x^3 + x^5) \), \( y = 0 \), \( x = 1 \), and \( x = 2 \).

35–40 Use the Endpaper Integral Table to evaluate the integral.

35. \( \int \sin 7x \cos 9x \, dx \)
36. \( \int (x^3 - x^2) e^{-x} \, dx \)
37. \( \int x \sqrt{x - 4} \, dx \)
38. \( \int \frac{dx}{x \sqrt{4x + 3}} \)
39. \( \int \tan^2 2x \, dx \)
40. \( \int \frac{3x - 1}{2 + x^2} \, dx \)

41–42 Approximate the integral using (a) the midpoint approximation \( M_{10} \), (b) the trapezoidal approximation \( T_{10} \), and (c) Simpson’s rule approximation \( S_{20} \). In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places.

41. \( \int_1^3 \frac{1}{\sqrt{x + 1}} \, dx \)
42. \( \int_{-1}^{1} \frac{1}{1 + x^2} \, dx \)

43–44 Use inequalities (12), (13), and (14) of Section 7.7 to find upper bounds on the errors in parts (a), (b), or (c) of the indicated exercise.

43. Exercise 41
44. Exercise 42

45–46 Use inequalities (12), (13), and (14) of Section 7.7 to find a number \( n \) of subintervals for (a) the midpoint approximation \( M_n \), (b) the trapezoidal approximation \( T_n \), and (c) Simpson’s rule approximation \( S_n \) to ensure the absolute error will be less than \( 10^{-4} \).

45. Exercise 41
46. Exercise 42

47–50 Evaluate the integral if it converges.

47. \( \int_0^{+\infty} e^{-x} \, dx \)
48. \( \int_{-\infty}^{0} \frac{dx}{x^2 + 4} \)
49. \( \int_0^{\pi/4} \frac{dx}{\sqrt{9 - x}} \)
50. \( \int_0^{1} \frac{1}{2x - 1} \, dx \)

51. Find the area that is enclosed between the \( x \)-axis and the curve \( y = (\ln x - 1)/x^2 \) for \( x \geq e \).

52. Find the volume of the solid that is generated when the region between the \( x \)-axis and the curve \( y = e^{-x} \) for \( x \geq 0 \) is revolved about the \( y \)-axis.

53. Find a positive value of \( a \) that satisfies the equation
   \[ \int_0^{+\infty} \frac{dx}{x^2 + a^2} \, dx = 1 \]

54. Consider the following methods for evaluating integrals:
   (a) \( \int x \sin x \, dx \)
   (b) \( \int \cos x \sin x \, dx \)

(cont.)
Chapter 7 Making Connections

1. Recall from Theorem 3.3.1 and the discussion preceding it that if \( f(x) > 0 \), then the function \( f \) is increasing and has an inverse function. Parts (a), (b), and (c) of this problem show that if this condition is satisfied and if \( f' \) is continuous, then a definite integral of \( f^{-1} \) can be expressed in terms of a definite integral of \( f \).

(a) Use integration by parts to show that
\[
\int_a^b f(x) \, dx = bf(b) - af(a) - \int_a^b xf'(x) \, dx
\]
(b) Use the result in part (a) to show that if \( y = f(x) \), then
\[
\int_a^b f(x) \, dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y) \, dy
\]
(c) Show that if we let \( \alpha = f(a) \) and \( \beta = f(b) \), then the result in part (b) can be written as
\[
\int_a^\beta f^{-1}(x) \, dx = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x) \, dx
\]

2. In each part, use the result in Exercise 1 to obtain the equation, and then confirm that the equation is correct by performing the integrations.

(a) \( \int_0^{\pi/2} \sin^{-1} x \, dx = \frac{1}{2} \sin^{-1} \left( \frac{1}{2} \right) - \int_0^{\pi/6} \sin x \, dx \)
(b) \( \int_2^e \ln x \, dx = (2e^2 - e) - \int_1^2 e^t \, dt \)

3. The Gamma function, \( \Gamma(x) \), is defined as
\[
\Gamma(x) = \int_0^\infty t^{x-1} \, e^{-t} \, dt
\]
It can be shown that this improper integral converges if and only if \( x > 0 \).

(a) Find \( \Gamma(1) \).
(b) Prove: \( \Gamma(x+1) = x\Gamma(x) \) for all \( x > 0 \). [Hint: Use integration by parts.]
(c) Use the results in parts (a) and (b) to find \( \Gamma(2) \), \( \Gamma(3) \), and \( \Gamma(4) \); and then make a conjecture about \( \Gamma(n) \) for positive integer values of \( n \).

(d) Show that \( \Gamma\left( \frac{1}{2} \right) = \sqrt{\pi} \). [Hint: See Exercise 66 of Section 7.8.]
(e) Use the results obtained in parts (a) and (d) to show that \( \Gamma\left( \frac{1}{2} \right) = \frac{1}{2} \sqrt{\pi} \) and \( \Gamma\left( \frac{3}{2} \right) = \frac{1}{2} \sqrt{\pi} \).

4. Refer to the Gamma function defined in Exercise 3 to show that

(a) \( \int_0^1 (\ln x)^n \, dx = (-1)^n \Gamma(n+1) \), \( n > 0 \) [Hint: Let \( t = - \ln x \).]
(b) \( \int_0^\infty e^{-x^n} \, dx = \Gamma\left( \frac{n+1}{n} \right) \), \( n > 0 \) [Hint: Let \( t = x^n \). Use the result in Exercise 3(b).]

5. A simple pendulum consists of a mass that swings in a vertical plane at the end of a massless rod of length \( L \), as shown in the accompanying figure. Suppose that a simple pendulum is displaced through an angle \( \theta_0 \) and released from rest. It can be
shown that in the absence of friction, the time \( T \) required for the pendulum to make one complete back-and-forth swing, called the *period*, is given by

\[
T = \sqrt{\frac{8L}{g}} \int_{0}^{0} \frac{1}{\sqrt{\cos \theta - \cos \theta_0}} \, d\theta
\]  

(1)

where \( \theta = \theta(t) \) is the angle the pendulum makes with the vertical at time \( t \). The improper integral in (1) is difficult to evaluate numerically. By a substitution outlined below it can be shown that the period can be expressed as

\[
T = 4 \sqrt{\frac{L}{g}} \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi
\]  

(2)

where \( k = \sin(\theta_0/2) \). The integral in (2) is called a *complete elliptic integral of the first kind* and is more easily evaluated by numerical methods.

(a) Obtain (2) from (1) by substituting

\[
\cos \theta = 1 - 2 \sin^2(\theta/2)
\]

\[
\cos \theta_0 = 1 - 2 \sin^2(\theta_0/2)
\]

\[
k = \sin(\theta_0/2)
\]

and then making the change of variable

\[
\sin \phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} = \frac{\sin(\theta/2)}{k}
\]

(b) Use (2) and the numerical integration capability of your CAS to estimate the period of a simple pendulum for which \( L = 1.5 \) ft, \( \theta_0 = 20^\circ \), and \( g = 32 \) ft/s\(^2\).

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**Expanding the Calculus Horizon**

To learn how numerical integration can be applied to the cost analysis of an engineering project, see the module entitled *Railroad Design* at: [www.wiley.com/college/anton](http://www.wiley.com/college/anton)
In the 1920s, excavation of an archeological site in Folsom, New Mexico, uncovered a collection of prehistoric stone spearheads now known as “Folsom points.” In 1950, carbon dating of charred bison bones found nearby confirmed that human hunters lived in the area between 9000 B.C. and 8000 B.C. We will study carbon dating in this chapter.

8.1 MODELING WITH DIFFERENTIAL EQUATIONS

In this section we will introduce some basic terminology and concepts concerning differential equations. We will also discuss the general idea of modeling with differential equations, and we will encounter important models that can be applied to demography, medicine, ecology, and physics. In later sections of this chapter we will investigate methods that may be used to solve these differential equations.

Table 8.1.1

<table>
<thead>
<tr>
<th>DIFFERENTIAL EQUATION</th>
<th>ORDER</th>
</tr>
</thead>
<tbody>
<tr>
<td>dy/dx = 3y</td>
<td>1</td>
</tr>
<tr>
<td>d²y/dx² - 6 dy/dx + 8y = 0</td>
<td>2</td>
</tr>
<tr>
<td>d³y/dt³ - t dy/dt + (t² - 1)y = eᵗ</td>
<td>3</td>
</tr>
<tr>
<td>y&quot; - y = e²x</td>
<td>1</td>
</tr>
<tr>
<td>y&quot; + y' = cos t</td>
<td>2</td>
</tr>
</tbody>
</table>

TERMINOLOGY

Recall from Section 5.2 that a differential equation is an equation involving one or more derivatives of an unknown function. In this section we will denote the unknown function by \( y = y(x) \) unless the differential equation arises from an applied problem involving time, in which case we will denote it by \( y = y(t) \). The order of a differential equation is the order of the highest derivative that it contains. Some examples are given in Table 8.1.1. The last two equations in that table are expressed in “prime” notation, which does not specify the independent variable explicitly. However, you will usually be able to tell from the equation itself or from the context in which it arises whether to interpret \( y' \) as \( dy/dx \) or \( dy/dt \).

SOLUTIONS OF DIFFERENTIAL EQUATIONS

A function \( y = y(x) \) is a solution of a differential equation on an open interval if the equation is satisfied identically on the interval when \( y \) and its derivatives are substituted...
into the equation. For example, \( y = e^{2x} \) is a solution of the differential equation

\[
\frac{dy}{dx} - y = e^{2x}
\]

(1)
on the interval \((-\infty, +\infty)\), since substituting \( y \) and its derivative into the left side of this equation yields

\[
\frac{dy}{dx} - y = \frac{d}{dx}[e^{2x}] - e^{2x} = 2e^{2x} - e^{2x} = e^{2x}
\]

for all real values of \( x \). However, this is not the only solution on \((-\infty, +\infty)\); for example, the function

\[
y = e^{2x} + Ce^x
\]

(2)
is also a solution for every real value of the constant \( C \), since

\[
\frac{dy}{dx} - y = \frac{d}{dx}[e^{2x} + Ce^x] - (e^{2x} + Ce^x) = (2e^{2x} + Ce^x) - (e^{2x} + Ce^x) = e^{2x}
\]

After developing some techniques for solving equations such as (1), we will be able to show that all solutions of (1) on \((-\infty, +\infty)\) can be obtained by substituting values for the constant \( C \) in (2). On a given interval, a solution of a differential equation from which all solutions on that interval can be derived by substituting values for arbitrary constants is called a **general solution** of the equation on the interval. Thus (2) is a general solution of (1) on the interval \((-\infty, +\infty)\).

The graph of a solution of a differential equation is called an **integral curve** for the equation, so the general solution of a differential equation produces a family of integral curves corresponding to the different possible choices for the arbitrary constants. For example, Figure 8.1.1 shows some integral curves for (1), which were obtained by assigning values to the arbitrary constant in (2).

**INITIAL-VALUE PROBLEMS**

When an applied problem leads to a differential equation, there are usually conditions in the problem that determine specific values for the arbitrary constants. As a rule of thumb, it requires \( n \) conditions to determine values for all \( n \) arbitrary constants in the general solution of an \( n \)th-order differential equation (one condition for each constant). For a first-order equation, the single arbitrary constant can be determined by specifying the value of the unknown function \( y(x) \) at an arbitrary \( x \)-value, say \( y(x_0) = y_0 \). This is called an **initial condition**, and the problem of solving a first-order equation subject to an initial condition is called a **first-order initial-value problem**. Geometrically, the initial condition \( y(x_0) = y_0 \) has the effect of isolating the integral curve that passes through the point \((x_0, y_0)\) from the complete family of integral curves.
8.1 Modeling with Differential Equations

Example 1

The solution of the initial-value problem
\[ \frac{dy}{dx} - y = e^{2x}, \quad y(0) = 3 \]

can be obtained by substituting the initial condition \( x = 0, y = 3 \) in the general solution (2) to find \( C \). We obtain
\[ 3 = e^0 + Ce^0 = 1 + C \]

Thus, \( C = 2 \), and the solution of the initial-value problem, which is obtained by substituting this value of \( C \) in (2), is
\[ y = e^{2x} + 2e^x \]

Geometrically, this solution is realized as the integral curve in Figure 8.1.1 that passes through the point \((0, 3)\).

Since many important principles in the physical and social sciences involve rates of change, it should not be surprising that such principles can often be modeled by differential equations. Here are some examples of the modeling process.

UNINHIBITED POPULATION GROWTH

One of the simplest models of population growth is based on the observation that when populations (people, plants, bacteria, and fruit flies, for example) are not constrained by environmental limitations, they tend to grow at a rate that is proportional to the size of the population—the larger the population, the more rapidly it grows.

To translate this principle into a mathematical model, suppose that \( y = y(t) \) denotes the population at time \( t \). At each point in time, the rate of increase of the population with respect to time is \( \frac{dy}{dt} \), so the assumption that the rate of growth is proportional to the population is described by the differential equation
\[ \frac{dy}{dt} = ky \quad (3) \]

where \( k \) is a positive constant of proportionality that can usually be determined experimentally. Thus, if the population is known at some point in time, say \( y = y_0 \) at time \( t = 0 \), then a formula for the population \( y(t) \) can be obtained by solving the initial-value problem
\[ \frac{dy}{dt} = ky, \quad y(0) = y_0 \]

INHIBITED POPULATION GROWTH; LOGISTIC MODELS

The uninhibited population growth model was predicated on the assumption that the population \( y = y(t) \) was not constrained by the environment. While this assumption is reasonable as long as the size of the population is relatively small, environmental effects become increasingly important as the population grows. In general, populations grow within ecological systems that can only support a certain number of individuals; the number \( L \) of such individuals is called the carrying capacity of the system. When \( y > L \), the population exceeds the capacity of the ecological system and tends to decrease toward \( L \); when \( y < L \), the population is below the capacity of the ecological system and tends to increase toward \( L \); when \( y = L \), the population is in balance with the capacity of the ecological system and tends to remain stable.

To translate this into a mathematical model, we must look for a differential equation in which \( y > 0, L > 0, \) and
\[ \frac{dy}{dt} < 0 \quad \text{if} \quad \frac{y}{L} > 1, \quad \frac{dy}{dt} > 0 \quad \text{if} \quad \frac{y}{L} < 1, \quad \frac{dy}{dt} = 0 \quad \text{if} \quad \frac{y}{L} = 1 \]
Moreover, when the population is far below the carrying capacity (i.e., \( \frac{y}{L} \approx 0 \)), then the environmental constraints should have little effect, and the growth rate should behave like the uninhibited population model. Thus, we want

\[
\frac{dy}{dt} \approx ky \quad \text{if} \quad \frac{y}{L} \approx 0
\]

A simple differential equation that meets all of these requirements is

\[
\frac{dy}{dt} = k \left( 1 - \frac{y}{L} \right) y
\]

where \( k \) is a positive constant of proportionality. Thus if \( k \) and \( L \) can be determined experimentally, and if the population is known at some point, say \( y(0) = y_0 \), then a formula for the population \( y(t) \) can be determined by solving the initial-value problem

\[
\frac{dy}{dt} = k \left( 1 - \frac{y}{L} \right) y, \quad y(0) = y_0 \quad (4)
\]

This theory of population growth is due to the Belgian mathematician P. F. Verhulst (1804–1849), who introduced it in 1838 and described it as “logistic growth.” Thus, the differential equation in (4) is called the logistic differential equation, and the growth model described by (4) is called the logistic model.

### Pharmacology

When a drug (say, penicillin or aspirin) is administered to an individual, it enters the bloodstream and then is absorbed by the body over time. Medical research has shown that the amount of a drug that is present in the bloodstream tends to decrease at a rate that is proportional to the amount of the drug present—the more of the drug that is present in the bloodstream, the more rapidly it is absorbed by the body.

To translate this principle into a mathematical model, suppose that \( y = y(t) \) is the amount of the drug present in the bloodstream at time \( t \). At each point in time, the rate of change in \( y \) with respect to \( t \) is \( \frac{dy}{dt} \), so the assumption that the rate of decrease is proportional to the amount \( y \) in the bloodstream translates into the differential equation

\[
\frac{dy}{dt} = -ky \quad (5)
\]

where \( k \) is a positive constant of proportionality that depends on the drug and can be determined experimentally. The negative sign is required because \( y \) decreases with time. Thus, if the initial dosage of the drug is known, say \( y = y_0 \) at time \( t = 0 \), then a formula for \( y(t) \) can be obtained by solving the initial-value problem

\[
\frac{dy}{dt} = -ky, \quad y(0) = y_0
\]

### Spread of Disease

Suppose that a disease begins to spread in a population of \( L \) individuals. Logic suggests that at each point in time the rate at which the disease spreads will depend on how many individuals are already affected and how many are not—as more individuals are affected, the opportunity to spread the disease tends to increase, but at the same time there are fewer individuals who are not affected, so the opportunity to spread the disease tends to decrease. Thus, there are two conflicting influences on the rate at which the disease spreads.

---

*Verhulst’s model fell into obscurity for nearly a hundred years because he did not have sufficient census data to test its validity. However, interest in the model was revived during the 1930s when biologists used it successfully to describe the growth of fruit fly and flour beetle populations. Verhulst himself used the model to predict that an upper limit of Belgium’s population would be approximately 9,400,000. In 2006 the population was about 10,379,000.*
8.1 Modeling with Differential Equations

To translate this into a mathematical model, suppose that \( y = y(t) \) is the number of individuals who have the disease at time \( t \), so of necessity the number of individuals who do not have the disease at time \( t \) is \( L - y \). As the value of \( y \) increases, the value of \( L - y \) decreases, so the conflicting influences of the two factors on the rate of spread \( dy/dt \) are taken into account by the differential equation

\[
\frac{dy}{dt} = ky(L - y)
\]

where \( k \) is a positive constant of proportionality that depends on the nature of the disease and the behavior patterns of the individuals and can be determined experimentally. Thus, if the number of affected individuals is known at some point in time, say \( y = y_0 \) at time \( t = 0 \), then a formula for \( y(t) \) can be obtained by solving the initial-value problem

\[
\frac{dy}{dt} = ky(L - y), \quad y(0) = y_0
\]

**Newton's Law of Cooling**

If a hot object is placed into a cool environment, the object will cool at a rate proportional to the difference in temperature between the object and the environment. Similarly, if a cold object is placed into a warm environment, the object will warm at a rate that is again proportional to the difference in temperature between the object and the environment. Together, these observations comprise a result known as **Newton's Law of Cooling**. (Newton’s Law of Cooling appeared previously in the exercises of Section 2.2 and was mentioned briefly in Section 5.8.) To translate this into a mathematical model, suppose that \( T = T(t) \) is the temperature of the object at time \( t \) and that \( T_e \) is the temperature of the environment, which is assumed to be constant. Since the rate of change \( dT/dt \) is proportional to \( T - T_e \), we have

\[
\frac{dT}{dt} = k(T - T_e)
\]

where \( k \) is a constant of proportionality. Moreover, since \( dT/dt \) is positive when \( T < T_e \), and is negative when \( T > T_e \), the sign of \( k \) must be negative. Thus if the temperature of the object is known at some time, say \( T = T_0 \) at time \( t = 0 \), then a formula for the temperature \( T(t) \) can be obtained by solving the initial-value problem

\[
\frac{dT}{dt} = k(T - T_e), \quad T(0) = T_0
\]

**Vibrations of Springs**

We conclude this section with an engineering model that leads to a second-order differential equation.

As shown in Figure 8.1.2, consider a block of mass \( m \) attached to the end of a horizontal spring. Assume that the block is then set into vibratory motion by pulling the spring beyond its natural position and releasing it at time \( t = 0 \). We will be interested in finding a mathematical model that describes the vibratory motion of the block over time.

To translate this problem into mathematical form, we introduce a horizontal \( x \)-axis whose positive direction is to the right and whose origin is at the right end of the spring when the spring is in its natural position (Figure 8.1.3). Our goal is to find a model for the coordinate \( x = x(t) \) of the point of attachment of the block to the spring as a function of time. In developing this model, we will assume that the only force on the mass \( m \) is the restoring force of the spring, and we will ignore the influence of other forces such as friction, air resistance, and so forth. Recall from Hooke’s Law (Section 6.6) that when the connection point has coordinate \( x(t) \), the restoring force is \(-kx(t)\), where \( k \) is the spring constant. [The negative sign is due to the fact that the restoring force is to the left when \( x(t) \) is positive, and the restoring force is to the right when \( x(t) \) is negative.] It follows from Newton’s
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Second Law of Motion [Equation (5) of Section 6.6] that this restoring force is equal to the product of the mass $m$ and the acceleration $d^2x/dt^2$ of the mass. In other words, we have

$$m \frac{d^2x}{dt^2} = -kx$$

which is a second-order differential equation for $x$. If at time $t = 0$ the mass is released from rest at position $x(0) = x_0$, then a formula for $x(t)$ can be found by solving the initial-value problem

$$m \frac{d^2x}{dt^2} = -kx, \quad x(0) = x_0, \quad x'(0) = 0 \quad (8)$$

[If at time $t = 0$ the mass is given an initial velocity $v_0 \neq 0$, then the condition $x'(0) = 0$ must be replaced by $x'(0) = v_0$.]

Quick Check Exercises 8.1  (See page 568 for answers.)

1. Match each differential equation with its family of solutions.
   (a) $x \frac{dy}{dx} = y$ (i) $y = x^2 + C$
   (b) $y'' = 4y$ (ii) $y = C_1 \sin 2x + C_2 \cos 2x$
   (c) $\frac{dy}{dx} = 2x$ (iii) $y = C_1e^{2x} + C_2e^{-2x}$
   (d) $\frac{d^2y}{dx^2} = -4y$ (iv) $y = Cx$

2. If $y = C_1e^{2x} + C_2xe^{2x}$ is the general solution of a differential equation, then the order of the equation is ________, and a solution to the differential equation that satisfies the initial conditions $y(0) = 1, y'(0) = 4$ is given by $y = ________$.

3. The graph of a differentiable function $y = y(x)$ passes through the point $(0, 1)$ and at every point $P(x, y)$ on the graph the tangent line is perpendicular to the line through $P$ and the origin. Find an initial-value problem whose solution is $y(x)$.

4. A glass of ice water with a temperature of $36^\circ F$ is placed in a room with a constant temperature of $68^\circ F$. Assuming that Newton’s Law of Cooling applies, find an initial-value problem whose solution is the temperature of water $t$ minutes after it is placed in the room. [Note: The differential equation will involve a constant of proportionality.]

Exercise Set 8.1

1. Confirm that $y = 3e^x$ is a solution of the initial-value problem $y' = 3x^2 y$, $y(0) = 3$.

2. Confirm that $y = \frac{1}{4}x^4 + 2 \cos x + 1$ is a solution of the initial-value problem $y' = x^3 - 2 \sin x$, $y(0) = 3$.

3–4 State the order of the differential equation, and confirm that the functions in the given family are solutions.
   3. (a) $(1 + x) \frac{dy}{dx} = y; \quad y = c(1 + x)$
   (b) $y'' + y = 0; \quad y = c_1 \sin t + c_2 \cos t$
   4. (a) $2 \frac{dy}{dx} + y = x - 1; \quad y = ce^{-x/2} + x - 3$
   (b) $y'' - y = 0; \quad y = c_1 e^t + c_2 e^{-t}$

5–8 True–False  Determine whether the statement is true or false. Explain your answer.
   5. The equation $\left( \frac{dy}{dx} \right)^2 = \frac{dy}{dx} + 2y$

   is an example of a second-order differential equation.

6. The differential equation

   $\frac{dy}{dx} = 2y + 1$

   has a solution that is constant.

7. We expect the general solution of the differential equation

   $\frac{d^3y}{dx^3} + 3 \frac{d^2y}{dx^2} - \frac{dy}{dx} + 4y = 0$

   to involve three arbitrary constants.

8. If every solution to a differential equation can be expressed in the form $y = Ae^{xt} + b$ for some choice of constants $A$ and $b$, then the differential equation must be of second order.

9–14 In each part, verify that the functions are solutions of the differential equation by substituting the functions into the equation.
   9. $y'' + y' - 2y = 0$
      (a) $e^{-1}x$ and $e^x$
      (b) $c_1 e^{-2x} + c_2 e^x$ (c, c constants)
10. $y'' - y' = 6y = 0$
   (a) $e^{-2x}$ and $e^{3x}$
   (b) $c_1 e^{-2x} + c_2 e^{3x}$ ($c_1, c_2$ constants)

11. $y'' - 4y' + 4y = 0$
   (a) $e^{2x}$ and $xe^{2x}$
   (b) $c_1 e^{2x} + c_2 xe^{2x}$ ($c_1, c_2$ constants)

12. $y'' - 8y' + 16y = 0$
   (a) $e^{4x}$ and $xe^{4x}$
   (b) $c_1 e^{4x} + c_2 xe^{4x}$ ($c_1, c_2$ constants)

13. $y'' + 4y = 0$
   (a) $sin 2x$ and $cos 2x$
   (b) $c_1 sin 2x + c_2 cos 2x$ ($c_1, c_2$ constants)

14. $y'' + 4y' + 13y = 0$
   (a) $e^{-2x}$ sin $3x$ and $e^{-2x}$ cos $3x$
   (b) $e^{-2x}(c_1 sin 3x + c_2 cos 3x)$ ($c_1, c_2$ constants)

15–20 Use the results of Exercises 9–14 to find a solution to the initial-value problem.

15. $y'' + y' - 2y = 0$, $y(0) = -1$, $y'(0) = -4$

16. $y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = 8$

17. $y'' - 4y' + 4y = 0$, $y(0) = 2$, $y'(0) = 2$

18. $y'' - 8y' + 16y = 0$, $y(0) = 1$, $y'(0) = 1$

19. $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$

20. $y'' + 4y' + 13y = 0$, $y(0) = -1$, $y'(0) = -1$

21–26 Find a solution to the initial-value problem.

21. $y' + 4x = 2$, $y(0) = 3$

22. $y'' + 6x = 0$, $y(0) = 1$, $y'(0) = 2$

23. $y' - y^2 = 0$, $y(1) = 2$ [Hint: Assume the solution has an inverse function $x = x(y)$. Find, and solve, a differential equation that involves $x(y)$]

24. $y' = 1 + y^2$, $y(0) = 0$ (See Exercise 23.)

25. $x^2y' + 2xy = 0$, $y(1) = 2$ [Hint: Interpret the left-hand side of the equation as the derivative of a product of two functions.]

26. $xy' + y = e^x$, $y(1) = 1 + e$ (See Exercise 25.)

### Focus on Concepts

27. (a) Suppose that a quantity $y = y(t)$ increases at a rate that is proportional to the square of the amount present, and suppose that at time $t = 0$, the amount present is $y_0$. Find an initial-value problem whose solution is $y(t)$.
   (b) Suppose that a quantity $y = y(t)$ decreases at a rate that is proportional to the square of the amount present, and suppose that at a time $t = 0$, the amount present is $y_0$. Find an initial-value problem whose solution is $y(t)$.

28. (a) Suppose that a quantity $y = y(t)$ changes in such a way that $dy/dt = k\sqrt{y}$, where $k > 0$. Describe how $y$ changes in words.

### 8.1 Modeling with Differential Equations

29. (a) Suppose that a particle moves along an $s$-axis in such a way that its velocity $v(t)$ is always half of $s(t)$. Find a differential equation whose solution is $s(t)$.
   (b) Suppose that an object moves along an $s$-axis in such a way that its acceleration $a(t)$ is always twice the velocity. Find a differential equation whose solution is $s(t)$.

30. Suppose that a body moves along an $s$-axis through a resistive medium in such a way that the velocity $v = v(t)$ decreases at a rate that is twice the square of the velocity.
   (a) Find a differential equation whose solution is the velocity $v(t)$.
   (b) Find a differential equation whose solution is the position $s(t)$.

31. Consider a solution $y = y(t)$ to the uninhibited population growth model.
   (a) Use Equation (3) to explain why $y$ will be an increasing function of $t$.
   (b) Use Equation (3) to explain why the graph $y = y(t)$ will be concave up.

32. Consider the logistic model for population growth.
   (a) Explain why there are two constant solutions to this model.
   (b) For what size of the population will the population be growing most rapidly?

33. Consider the model for the spread of disease.
   (a) Explain why there are two constant solutions to this model.
   (b) For what size of the infected population is the disease spreading most rapidly?

34. Explain why there is exactly one constant solution to the Newton’s Law of Cooling model.

35. Show that if $c_1$ and $c_2$ are any constants, the function
   
   $x = x(t) = c_1 cos \left( \frac{k}{m} t \right) + c_2 sin \left( \frac{k}{m} t \right)$

   is a solution to the differential equation for the vibrating spring. (The corresponding motion of the spring is referred to as simple harmonic motion.)

36. (a) Use the result of Exercise 35 to solve the initial-value problem in (8).
   (b) Find the amplitude, period, and frequency of your answer to part (a), and interpret each of these in terms of the motion of the spring.

37. Writing Select one of the models in this section and write a paragraph that discusses conditions under which the model would not be appropriate. How might you modify the model to take those conditions into account?
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✔  QUICK CHECK ANSWERS 8.1

1. (a) (iv) (b) (iii) (c) (i) (d) (ii)  2. \( e^{2x} + 2xe^{2x} \)  3. \( \frac{dy}{dx} = -\frac{x}{y}, \ y(0) = 1 \)  4. \( \frac{dT}{dt} = k(T - 68), \ T(0) = 36 \)

8.2  SEPARATION OF VARIABLES

In this section we will discuss a method, called “separation of variables,” that can be used to solve a large class of first-order differential equations of a particular form. We will use this method to investigate mathematical models for exponential growth and decay, including population models and carbon dating.

First-Order Separable Equations

We will now consider a method of solution that can often be applied to first-order equations that are expressible in the form

\[ h(y) \frac{dy}{dx} = g(x) \]

Such first-order equations are said to be separable. Some examples of separable equations are given in Table 8.2.1. The name “separable” arises from the fact that Equation (1) can be rewritten in the differential form

\[ h(y) \, dy = g(x) \, dx \]

in which the expressions involving \( x \) and \( y \) appear on opposite sides. The process of rewriting (1) in form (2) is called separating variables.

Table 8.2.1

<table>
<thead>
<tr>
<th>Equation</th>
<th>Form (1)</th>
<th>( h(y) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dy}{dx} = \frac{x}{y} )</td>
<td>( y \frac{dy}{dx} = x )</td>
<td>( y )</td>
<td>( x )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = x^2y^3 )</td>
<td>( \frac{1}{y^3} \frac{dy}{dx} = \frac{1}{x^2} )</td>
<td>( \frac{1}{x^2} )</td>
<td>( y^3 )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = y )</td>
<td>( \frac{1}{y} \frac{dy}{dx} = 1 )</td>
<td>( 1 )</td>
<td>( \frac{1}{y} )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = y - \frac{y}{x} )</td>
<td>( \frac{1}{y} \frac{dy}{dx} = 1 - \frac{1}{x} )</td>
<td>( \frac{1}{y} )</td>
<td>( 1 - \frac{1}{x} )</td>
</tr>
</tbody>
</table>

To motivate a method for solving separable equations, assume that \( h(y) \) and \( g(x) \) are continuous functions of their respective variables, and let \( H(y) \) and \( G(x) \) denote antiderivatives of \( h(y) \) and \( g(x) \), respectively. Consider the equation that results if we integrate both sides of (2), the left side with respect to \( y \) and the right side with respect to \( x \). We then have

\[ \int h(y) \, dy = \int g(x) \, dx \]

or, equivalently,

\[ H(y) = G(x) + C \]

where \( C \) denotes a constant. We claim that a differentiable function \( y = y(x) \) is a solution to (1) if and only if \( y \) satisfies Equation (4) for some choice of the constant \( C \).
Suppose that \( y = y(x) \) is a solution to (1). It then follows from the chain rule that
\[
\frac{d}{dx}[H(y)] = \frac{dH}{dy} \frac{dy}{dx} = h(y) \frac{dy}{dx} = g(x) = \frac{dG}{dx}
\]
(5)

Since the functions \( H(y) \) and \( G(x) \) have the same derivative with respect to \( x \), they must differ by a constant (Theorem 4.8.3). It then follows that \( y \) satisfies (4) for an appropriate choice of \( C \). Conversely, if \( y = y(x) \) is defined implicitly by Equation (4), then implicit differentiation shows that (5) is satisfied, and thus \( y(x) \) is a solution to (1) (Exercise 67).

Because of this, it is common practice to refer to Equation (4) as the “solution” to (1).

In summary, we have the following procedure for solving (1), called \textit{separation of variables}:

\begin{itemize}
  \item \textbf{Step 1.} Separate the variables in (1) by rewriting the equation in the differential form
  \[ h(y) \, dy = g(x) \, dx \]
  \item \textbf{Step 2.} Integrate both sides of the equation in Step 1 (the left side with respect to \( y \) and the right side with respect to \( x \)):
  \[ \int h(y) \, dy = \int g(x) \, dx \]
  \item \textbf{Step 3.} If \( H(y) \) is any antiderivative of \( h(y) \) and \( G(x) \) is any antiderivative of \( g(x) \), then the equation
  \[ H(y) = G(x) + C \]
  will generally define a family of solutions implicitly. In some cases it may be possible to solve this equation explicitly for \( y \).
\end{itemize}

\textbf{Example 1} \hspace{1em} Solve the differential equation
\[ \frac{dy}{dx} = -4xy^2 \]
and then solve the initial-value problem
\[ \frac{dy}{dx} = -4xy^2, \quad y(0) = 1 \]

\textbf{Solution.} For \( y \neq 0 \) we can write the differential equation in form (1) as
\[ \frac{1}{y^2} \frac{dy}{dx} = -4x \]
Separating variables and integrating yields
\[ \int \frac{1}{y^2} \, dy = \int -4x \, dx \]
or
\[ \frac{1}{y} = -2x^2 + C \]
Solving for \( y \) as a function of \( x \), we obtain
\[ y = \frac{1}{2x^2 - C} \]
The initial condition \( y(0) = 1 \) requires that \( y = 1 \) when \( x = 0 \). Substituting these values into our solution yields \( C = -1 \) (verify). Thus, a solution to the initial-value problem is

\[
y = \frac{1}{2x^2 + 1}
\]  

(6)

Some integral curves and our solution of the initial-value problem are graphed in Figure 8.2.1.

One aspect of our solution to Example 1 deserves special comment. Had the initial condition been \( y(0) = 0 \) instead of \( y(0) = 1 \), the method we used would have failed to yield a solution to the resulting initial-value problem (Exercise 25). This is due to the fact that we assumed \( y \neq 0 \) in order to rewrite the equation \( \frac{dy}{dx} = -4xy^2 \) in the form

\[
\frac{1}{y^2} \frac{dy}{dx} = -4x
\]

It is important to be aware of such assumptions when manipulating a differential equation algebraically.

**Example 2** Solve the initial-value problem

\[
(4y - \cos y) \frac{dy}{dx} - 3x^2 = 0, \quad y(0) = 0
\]

**Solution.** We can write the differential equation in form (1) as

\[
(4y - \cos y) \frac{dy}{dx} = 3x^2
\]

Separating variables and integrating yields

\[
(4y - \cos y) \, dy = 3x^2 \, dx
\]

\[
\int (4y - \cos y) \, dy = \int 3x^2 \, dx
\]

or

\[
2y^2 - \sin y = x^3 + C
\]  

(7)

For the initial-value problem, the initial condition \( y(0) = 0 \) requires that \( y = 0 \) if \( x = 0 \). Substituting these values into (7) to determine the constant of integration yields \( C = 0 \) (verify). Thus, the solution of the initial-value problem is

\[
2y^2 - \sin y = x^3
\]

or

\[
x = \frac{3}{2} \sqrt{2y^2 - \sin y}
\]  

(8)

Some integral curves and the solution of the initial-value problem in Example 2 are graphed in Figure 8.2.2.

Initial-value problems often result from geometrical questions, as in the following example.

**Example 3** Find a curve in the \( xy \)-plane that passes through \((0, 3)\) and whose tangent line at a point \((x, y)\) has slope \(2x/y^3\).
8.2 Separation of Variables

**Solution.** Since the slope of the tangent line is $dy/dx$, we have

$$\frac{dy}{dx} = \frac{2x}{y^2} \quad (9)$$

and, since the curve passes through $(0, 3)$, we have the initial condition

$$y(0) = 3$$

Equation (9) is separable and can be written as

$$y^2 \, dy = 2x \, dx$$

so

$$\int y^2 \, dy = \int 2x \, dx \quad \text{or} \quad \frac{1}{3}y^3 = x^2 + C$$

It follows from the initial condition that $y = 3$ if $x = 0$. Substituting these values into the last equation yields $C = 9$ (verify), so the equation of the desired curve is

$$\frac{1}{3}y^3 = x^2 + 9 \quad \text{or} \quad y = (3x^2 + 27)^{1/3} \quad \blacksquare$$

**Exponential Growth and Decay Models**

The population growth and pharmacology models developed in Section 8.1 are examples of a general class of models called exponential models. In general, exponential models arise in situations where a quantity increases or decreases at a rate that is proportional to the amount of the quantity present. More precisely, we make the following definition.

**8.2.1 Definition** A quantity $y = y(t)$ is said to have an exponential growth model if it increases at a rate that is proportional to the amount of the quantity present, and it is said to have an exponential decay model if it decreases at a rate that is proportional to the amount of the quantity present. Thus, for an exponential growth model, the quantity $y(t)$ satisfies an equation of the form

$$\frac{dy}{dt} = ky \quad (k > 0) \quad (10)$$

and for an exponential decay model, the quantity $y(t)$ satisfies an equation of the form

$$\frac{dy}{dt} = -ky \quad (k > 0) \quad (11)$$

The constant $k$ is called the growth constant or the decay constant, as appropriate.

Equations (10) and (11) are separable since they have the form of (1), but with $t$ rather than $x$ as the independent variable. To illustrate how these equations can be solved, suppose that a positive quantity $y = y(t)$ has an exponential growth model and that we know the amount of the quantity at some point in time, say $y = y_0$ when $t = 0$. Thus, a formula for $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

Separating variables and integrating yields

$$\int \frac{1}{y} \, dy = \int k \, dt$$

or (since $y > 0$)

$$\ln y = kt + C \quad \text{(12)}$$
The initial condition implies that $y = y_0$ when $t = 0$. Substituting these values in (12) yields $C = \ln y_0$ (verify). Thus,

$$\ln y = kt + \ln y_0$$

from which it follows that

$$y = e^{\ln y} = e^{kt+\ln y_0}$$

or, equivalently,

$$y = y_0e^{kt}$$

We leave it for you to show that if $y = y(t)$ has an exponential decay model, and if $y(0) = y_0$, then

$$y = y_0e^{-kt}$$

\[\text{(13)}\]

**INTERPRETING THE GROWTH AND DECAY CONSTANTS**

The significance of the constant $k$ in Formulas (13) and (14) can be understood by reexamining the differential equations that gave rise to these formulas. For example, in the case of the exponential growth model, Equation (10) can be rewritten as

$$k = \frac{dy}{dt}$$

which states that the growth rate as a fraction of the entire population remains constant over time, and this constant is $k$. For this reason, $k$ is called the *relative growth rate* of the population. It is usual to express the relative growth rate as a percentage. Thus, a relative growth rate of 3% per unit of time in an exponential growth model means that $k = 0.03$. Similarly, the constant $k$ in an exponential decay model is called the *relative decay rate*.

**Example 4**

According to the U.S. Census Bureau, the world population in 2011 was 6.9 billion and growing at a rate of about 1.10% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2030.

**Solution.** We assume that the population at the beginning of 2011 was 6.9 billion and let

$t = \text{time elapsed from the beginning of 2011 (in years)}$

$y = \text{world population (in billions)}$

Since the beginning of 2011 corresponds to $t = 0$, it follows from the given data that

$y_0 = y(0) = 6.9 \text{ billion}$

Since the growth rate is 1.10% ($k = 0.011$), it follows from (13) that the world population at time $t$ will be

$$y(t) = y_0e^{kt} = 6.9e^{0.011t}$$

(16)

Since the beginning of the year 2030 corresponds to an elapsed time of $t = 19$ years ($2030 - 2011 = 19$), it follows from (16) that the world population by the year 2030 will be

$$y(19) = 6.9e^{0.011(19)} \approx 8.5$$

which is a population of approximately 8.5 billion.

**DOUBLING TIME AND HALF-LIFE**

If a quantity $y$ has an exponential growth model, then the time required for the original size to double is called the *doubling time*, and if $y$ has an exponential decay model, then the time required for the original size to reduce by half is called the *half-life*. As it turns out, doubling time and half-life depend only on the growth or decay rate and not on the amount present initially. To see why this is so, suppose that $y = y(t)$ has an exponential growth model

$$y = y_0e^{kt}$$

(17)
and let $T$ denote the amount of time required for $y$ to double in size. Thus, at time $t = T$
the value of $y$ will be $2y_0$, and hence from (17)

$$2y_0 = y_0 e^{kT} \quad \text{or} \quad e^{kT} = 2$$

Taking the natural logarithm of both sides yields $kT = \ln 2$, which implies that the doubling

time is

$$T = \frac{1}{k} \ln 2$$  \hspace{1cm} (18)

We leave it as an exercise to show that Formula (18) also gives the half-life of an exponential decay model. Observe that this formula does not involve the initial amount $y_0$, so that in an exponential growth or decay model, the quantity $y$ doubles (or reduces by half) every $T$ units (Figure 8.2.3).

**Example 5**  It follows from (18) that with a continued growth rate of 1.10% per year,
the doubling time for the world population will be

$$T = \frac{1}{0.011} \ln 2 \approx 63$$

or approximately 63 years. Thus, with a continued 1.10% annual growth rate the population
of 6.9 billion in 2011 will double to 13.8 billion by the year 2074 and will double again to
27.6 billion by 2137.

**RADIOACTIVE DECAY**

It is a fact of physics that radioactive elements disintegrate spontaneously in a process
called radioactive decay. Experimentation has shown that the rate of disintegration is
proportional to the amount of the element present, which implies that the amount $y = y(t)$
of a radioactive element present as a function of time has an exponential decay model.

Every radioactive element has a specific half-life; for example, the half-life of radioactive
carbon-14 is about 5730 years. Thus, from (18), the decay constant for this element is

$$k = \frac{1}{T} \ln 2 = \frac{\ln 2}{5730} \approx 0.000121$$

and this implies that if there are $y_0$ units of carbon-14 present at time $t = 0$, then the number
of units present after $t$ years will be approximately

$$y(t) = y_0 e^{-0.000121 t}$$  \hspace{1cm} (19)

**Example 6**  If 100 grams of radioactive carbon-14 are stored in a cave for 1000 years,
how many grams will be left at that time?

**Solution.**  From (19) with $y_0 = 100$ and $t = 1000$, we obtain

$$y(1000) = 100e^{-0.000121(1000)} = 100e^{-0.121} \approx 88.6$$

Thus, about 88.6 grams will be left.

**CARBON DATING**

When the nitrogen in the Earth’s upper atmosphere is bombarded by cosmic radiation,
the radioactive element carbon-14 is produced. This carbon-14 combines with oxygen to
form carbon dioxide, which is ingested by plants, which in turn are eaten by animals. In
this way all living plants and animals absorb quantities of radioactive carbon-14. In 1947
the American nuclear scientist W. F. Libby\(^7\) proposed the theory that the percentage of

carbon-14 in the atmosphere and in living tissues of plants is the same. When a plant or animal dies, the carbon-14 in the tissue begins to decay. Thus, the age of an artifact that contains plant or animal material can be estimated by determining what percentage of its original carbon-14 content remains. Various procedures, called carbon dating or carbon-14 dating, have been developed for measuring this percentage.

**Example 7** In 1988 the Vatican authorized the British Museum to date a cloth relic known as the Shroud of Turin, possibly the burial shroud of Jesus of Nazareth. This cloth, which first surfaced in 1356, contains the negative image of a human body that was widely believed to be that of Jesus. The report of the British Museum showed that the fibers in the cloth contained between 92% and 93% of their original carbon-14. Use this information to estimate the age of the shroud.

**Solution.** From (19), the fraction of the original carbon-14 that remains after \( t \) years is

\[
y(t) = \frac{y_0}{e^{-0.000121t}}
\]

Taking the natural logarithm of both sides and solving for \( t \), we obtain

\[
t = \frac{-1}{0.000121} \ln \left( \frac{y(t)}{y_0} \right)
\]

Thus, taking \( y(t)/y_0 \) to be 0.93 and 0.92, we obtain

\[
t = \frac{-1}{0.000121} \ln(0.93) \approx 600
\]

\[
t = \frac{-1}{0.000121} \ln(0.92) \approx 689
\]

This means that when the test was done in 1988, the shroud was between 600 and 689 years old, thereby placing its origin between 1299 A.D. and 1388 A.D. Thus, if one accepts the validity of carbon-14 dating, the Shroud of Turin cannot be the burial shroud of Jesus of Nazareth.

**Quick Check Exercises 8.2** (See page 579 for answers.)

1. Solve the first-order separable equation

\[
h(y) \frac{dy}{dx} = g(x)
\]

by completing the following steps:

**Step 1.** Separate the variables by writing the equation in the differential form ____

**Step 2.** Integrate both sides of the equation in Step 1: ____

**Step 3.** If \( H(y) \) is any antiderivative of \( h(y) \), \( G(x) \) is any antiderivative of \( g(x) \), and \( C \) is an unspecified constant, then, as suggested by Step 2, the equation ____ will generally define a family of solutions to \( h(y) \frac{dy}{dx} = g(x) \) implicitly.

2. Suppose that a quantity \( y = y(t) \) has an exponential growth model with growth constant \( k > 0 \).

   (a) \( y(t) \) satisfies a first-order differential equation of the form \( \frac{dy}{dt} = \) ____

   (b) In terms of \( k \), the doubling time of the quantity is ____

   (c) If \( y_0 = y(0) \) is the initial amount of the quantity, then an explicit formula for \( y(t) \) is given by \( y(t) = \) ____

3. Suppose that a quantity \( y = y(t) \) has an exponential decay model with decay constant \( k > 0 \).

   (a) \( y(t) \) satisfies a first-order differential equation of the form \( \frac{dy}{dt} = \) ____

   (b) In terms of \( k \), the half-life of the quantity is ____

   (c) If \( y_0 = y(0) \) is the initial amount of the quantity, then an explicit formula for \( y(t) \) is given by \( y(t) = \) ____.
4. The initial-value problem
\[ \frac{dy}{dx} = \frac{-x}{y}, \quad y(0) = 1 \]
has solution \( y(x) = \).\\

EXERCISE SET 8.2  

1–10 Solve the differential equation by separation of variables. Where reasonable, express the family of solutions as explicit functions of \( x \).

1. \( \frac{dy}{dx} = \frac{y}{x} \)

2. \( \frac{dy}{dx} = 2(1 + y^2)x \)

3. \( \frac{\sqrt{1 + x^2} \, dy}{1 + y} = -x \)

4. \( (1 + x^4) \frac{dy}{dx} = \frac{x^3}{y} \)

5. \( (2 + 2y^2)y' = e^y \)

6. \( y' = -xy \)

7. \( e^{-y} \sin x - y \cos^2 x = 0 \)

8. \( y' - (1 + x)(1 + y^2) = 0 \)

9. \( \frac{dy}{dx} - \frac{y^2 - y}{\sin x} = 0 \)

10. \( y - \frac{dy}{dx} \sec x = 0 \)

11–14 Solve the initial-value problem by separation of variables.

11. \( y' = \frac{3x^2}{2y + \cos y}, \quad y(0) = \pi \)

12. \( y' - xe^y = 2e^y, \quad y(0) = 0 \)

13. \( \frac{dy}{dt} = \frac{2t + 1}{2y - 2}, \quad y(0) = -1 \)

14. \( y' \cosh^2 x - y \cosh 2x = 0, \quad y(0) = 3 \)

15. (a) Sketch some typical integral curves of the differential equation \( y' = y/2x \).

(b) Find an equation for the integral curve that passes through the point (2, 1).

16. (a) Sketch some typical integral curves of the differential equation \( y' = -x/y \).

(b) Find an equation for the integral curve that passes through the point (3, 4).

17–18 Solve the differential equation and then use a graphing utility to generate five integral curves for the equation.

17. \( (x^2 + 4) \frac{dy}{dx} + xy = 0 \)

18. \( (\cos y) y' = \cos x \)

19–20 Solve the differential equation. If you have a CAS with implicit plotting capability, use the CAS to generate five integral curves for the equation.

19. \( y' = \frac{x^2}{1 - y^2} \)

20. \( y' = \frac{y}{1 + y^2} \)

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. Every differential equation of the form \( y' = f(y) \) is separable.

22. A differential equation of the form \( h(x) \frac{dy}{dx} = g(y) \) is not separable.

23. If a radioactive element has a half-life of 1 minute, and if a container holds 32 g of the element at 1:00 p.m., then the amount remaining at 1:05 p.m. will be 1 g.

24. If a population is growing exponentially, then the time it takes the population to quadruple is independent of the size of the population.

25. Suppose that the initial condition in Example 1 had been \( y(0) = 0 \). Show that none of the solutions generated in Example 1 satisfy this initial condition, and then solve the initial-value problem

\[ \frac{dy}{dx} = -4xy^2, \quad y(0) = 0 \]

Why does the method of Example 1 fail to produce this particular solution?

26. Find all ordered pairs \((x_0, y_0)\) such that if the initial condition in Example 1 is replaced by \( y(x_0) = y_0 \), the solution of the resulting initial-value problem is defined for all real numbers.

27. Find an equation of a curve with \( x\)-intercept 2 whose tangent line at any point \((x, y)\) has slope \( xe^{-y} \).

28. Use a graphing utility to generate a curve that passes through the point (1, 1) and whose tangent line at \((x, y)\) is perpendicular to the line through \((x, y)\) with slope \(-2y/(3x^2)\).

29. Suppose that an initial population of 10,000 bacteria grows exponentially at a rate of 2% per hour and that \( y = y(t) \) is the number of bacteria present \( t \) hours later.

(a) Find an initial-value problem whose solution is \( y(t) \).

(b) Find a formula for \( y(t) \).

(c) How long does it take for the initial population of bacteria to double?

(d) How long does it take for the population of bacteria to reach 45,000?

30. A cell of the bacterium \( E. \ coli \) divides into two cells every 20 minutes when placed in a nutrient culture. Let \( y = y(t) \) be the number of cells that are present \( t \) minutes after a single cell is placed in the culture. Assume that the growth of the bacteria is approximated by an exponential growth model.

(a) Find an initial-value problem whose solution is \( y(t) \).

(b) Find a formula for \( y(t) \).
31. Radon-222 is a radioactive gas with a half-life of 3.83 days. This gas is a health hazard because it tends to get trapped in the basements of houses, and many health officials suggest that homeowners seal their basements to prevent entry of the gas. Assume that $5.0 \times 10^7$ radon atoms are trapped in a basement at the time it is sealed and that $y(t)$ is the number of atoms present $t$ days later.
   (a) Find an initial-value problem whose solution is $y(t)$.
   (b) Find a formula for $y(t)$.
   (c) How many atoms will be present after 30 days?
   (d) How long will it take for 90% of the original quantity of gas to decay?

32. Polonium-210 is a radioactive element with a half-life of 140 days. Assume that 10 milligrams of the element are placed in a lead container and that $y(t)$ is the number of milligrams present $t$ days later.
   (a) Find an initial-value problem whose solution is $y(t)$.
   (b) Find a formula for $y(t)$.
   (c) How many milligrams will be present after 10 weeks?
   (d) How long will it take for 70% of the original sample to decay?

33. Suppose that 100 fruit flies are placed in a breeding container that can support at most 10,000 flies. Assuming that the population grows exponentially at a rate of 2% per day, how long will it take for the container to reach capacity?

34. Suppose that the town of Grayrock had a population of 10,000 in 2006 and a population of 12,000 in 2011. Assuming an exponential growth model, in what year will the population reach 20,000?

35. A scientist wants to determine the half-life of a certain radioactive substance. She determines that in exactly 5 days a 10.0-milligram sample of the substance decays to 3.5 milligrams. Based on these data, what is the half-life?

36. Suppose that 30% of a certain radioactive substance decays in 5 years.
   (a) What is the half-life of the substance in years?
   (b) Suppose that a certain quantity of this substance is stored in a cave. What percentage of it will remain after $t$ years?

37. (a) Make a conjecture about the effect on the graphs of $y = y_0 e^{kt}$ and $y = y_0 e^{-kt}$ of varying $k$ and keeping $y_0$ fixed. Confirm your conjecture with a graphing utility.
   (b) Make a conjecture about the effect on the graphs of $y = y_0 e^{kt}$ and $y = y_0 e^{-kt}$ of varying $y_0$ and keeping $k$ fixed. Confirm your conjecture with a graphing utility.

38. (a) What effect does increasing $y_0$ and keeping $k$ fixed have on the doubling time or half-life of an exponential model? Justify your answer.
   (b) What effect does increasing $k$ and keeping $y_0$ fixed have on the doubling time and half-life of an exponential model? Justify your answer.

39. (a) There is a trick, called the Rule of 70, that can be used to get a quick estimate of the doubling time or half-life of an exponential model. According to this rule, the doubling time or half-life is roughly 70 divided by the percentage growth or decay rate. For example, we showed in Example 5 that with a continued growth rate of 1.10% per year the world population would double every 63 years. This result agrees with the Rule of 70, since 70/1.10 ≈ 63.6. Explain why this rule works.
   (b) Use the Rule of 70 to estimate the doubling time of a population that grows exponentially at a rate of 1% per year.
   (c) Use the Rule of 70 to estimate the half-life of a population that decreases exponentially at a rate of 3.5% per hour.
   (d) Use the Rule of 70 to estimate the growth rate that would be required for a population growing exponentially to double every 10 years.

40. Find a formula for the tripling time of an exponential growth model.

41. In 1950, a research team digging near Folsom, New Mexico, found charred bison bones along with some leaf-shaped projectile points (called the “Folsom points”) that had been made by a Paleo-Indian hunting culture. It was clear from the evidence that the bison had been cooked and eaten by the makers of the points, so that carbon-14 dating of the bones made it possible for the researchers to determine when the hunters roamed North America. Tests showed that the bones contained between 27% and 30% of their original carbon-14. Use this information to show that the hunters lived roughly between 9000 B.C. and 8000 B.C.

42. (a) Use a graphing utility to make a graph of $p_{	ext{rem}}$ versus $t$, where $p_{	ext{rem}}$ is the percentage of carbon-14 that remains in an artifact after $t$ years.
   (b) Use the graph to estimate the percentage of carbon-14 that would have to have been present in the 1988 test of the Shroud of Turin for it to have been the burial shroud of Jesus of Nazareth (see Example 7).

43. (a) It is currently accepted that the half-life of carbon-14 might vary ±40 years from its nominal value of 5730 years. Does this variation make it possible that the Shroud of Turin dates to the time of Jesus of Nazareth (see Example 7)?
   (b) Review the subsection of Section 3.5 entitled Error Propagation, and then estimate the percentage error that
44. Suppose that a quantity $y$ has an exponential growth model $y = y_0 e^{kt}$ or an exponential decay model $y = y_0 e^{-kt}$, and it is known that $y = y_1$ if $t = t_1$. In each case find a formula for $k$ in terms of $y_0$, $y_1$, and $t_1$, assuming that $t_1 \neq 0$.

45. (a) Show that if a quantity $y = y(t)$ has an exponential model, and if $y(t_1) = y_1$ and $y(t_2) = y_2$, then the doubling time or the half-life $T$ is
\[ T = \frac{(t_2 - t_1) \ln 2}{\ln(y_2/y_1)}. \]

(b) In a certain 1-hour period the number of bacteria in a colony increases by 25%. Assuming an exponential growth model, what is the doubling time for the colony?

46. Suppose that $P$ dollars is invested at an annual interest rate of $r \times 100\%$. If the accumulated interest is credited to the account at the end of the year, then the interest is said to be compounded annually; if it is credited at the end of each 6-month period, then it is said to be compounded semiannually; and if it is credited at the end of each 3-month period, then it is said to be compounded quarterly. The more frequently the interest is compounded, the better it is for the investor since more of the interest is itself earning interest.

(a) Show that if interest is compounded $n$ times a year at equally spaced intervals, then the value $A$ of the investment after $t$ years is
\[ A = P \left(1 + \frac{r}{n}\right)^{nt}. \]

(b) One can imagine interest to be compounded each day, each hour, each minute, and so forth. Carried to the limit one can conceive of interest compounded at each instant of time; this is called continuous compounding. Thus, from part (a), the value $A$ of $P$ dollars after $t$ years when invested at an annual rate of $r \times 100\%$, compounded continuously, is
\[ A = \lim_{n \to +\infty} P \left(1 + \frac{r}{n}\right)^{nt}. \]

Use the fact that $\lim_{x \to 0} (1 + x)^{1/x} = e$ to prove that $A = Pe^{rt}$.

(c) Use the result in part (b) to show that money invested at continuous compound interest increases at a rate proportional to the amount present.

47. (a) If $1000 is invested at 8% per year compounded continuously (Exercise 46), what will the investment be worth after 5 years?

(b) If it is desired that an investment at 8% per year compounded continuously should have a value of $10,000 after 10 years, how much should be invested now?

(c) How long does it take for an investment at 8% per year compounded continuously to double in value?

8.2 Separation of Variables

48. What is the effective annual interest rate for an interest rate of $r\%$ per year compounded continuously?

49. Assume that $y = y(t)$ satisfies the logistic equation with $y_0 = y(0)$ the initial value of $y$.

(a) Use separation of variables to derive the solution
\[ y = \frac{y_0}{1 + (L - y_0)e^{-kt}}. \]

(b) Use part (a) to show that $\lim_{t \to +\infty} y(t) = L$.

50. Use your answer to Exercise 49 to derive a solution to the model for the spread of disease [Equation (6) of Section 8.1].

51. The graph of a solution to the logistic equation is known as a logistic curve, and if $y_0 > 0$, it has one of four general shapes, depending on the relationship between $y_0$ and $L$. In each case, assume that $k = 1$ and use a graphing utility to plot a logistic curve satisfying the given condition.

(a) $y_0 > L$

(b) $y_0 = L$

(c) $L/2 \leq y_0 < L$

(d) $0 < y_0 < L/2$

52–53 The graph of a logistic model
\[ y = \frac{y_0L}{y_0 + (L - y_0)e^{-kt}} \]
is shown. Estimate $y_0$, $L$, and $k$.

54. Plot a solution to the initial-value problem
\[ \frac{dy}{dt} = 0.98 \left(1 - \frac{y}{5}\right), \quad y_0 = 1. \]

55. Suppose that the growth of a population $y = y(t)$ is given by the logistic equation
\[ y = \frac{60}{5 + 7e^{-0.5t}} \]

(a) What is the population at time $t = 0$?

(b) What is the carrying capacity $L$?

(c) What is the constant $k$?

(d) When does the population reach half of the carrying capacity?

(e) Find an initial-value problem whose solution is $y(t)$.

56. Suppose that the growth of a population $y = y(t)$ is given by the logistic equation
\[ y = \frac{1000}{1 + 999e^{-0.9t}} \]

(a) What is the population at time $t = 0$?

(b) What is the carrying capacity $L$?

(c) What is the constant $k$? (cont.)
57. Suppose that a university residence hall houses 1000 students. Following the semester break, 20 students in the hall return with the flu, and 5 days later 35 students have the flu.
(a) Use the result of Exercise 50 to find the number of students who will have the flu after returning to school.
(b) Make a table that illustrates how the flu spreads day to day over a 2-week period.
(c) Use a graphing utility to generate a graph that illustrates how the flu spreads over a 2-week period.
(d) When does the population reach 75% of the carrying capacity?
(e) Find an initial-value problem whose solution is $y(t)$. 

58. Suppose that at time $t = 0$ an object with temperature $T_0$ is placed in a room with constant temperature $T_a$. If $T_0 < T_a$, then the temperature of the object will increase, and if $T_0 > T_a$, then the temperature will decrease. Assuming that Newton’s Law of Cooling applies, show that in both cases the temperature $T(t)$ at time $t$ is given by

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

where $k$ is a positive constant.

59. A cup of water with a temperature of 95°C is placed in a room with a constant temperature of 21°C.
(a) Assuming that Newton’s Law of Cooling applies, use the result of Exercise 58 to find the temperature of the water $t$ minutes after it is placed in the room. [Note: The solution will involve a constant of proportionality.]
(b) How many minutes will it take for the water to reach a temperature of 51°C if it cools to 85°C in 1 minute?

60. A glass of lemonade with a temperature of 40°F is placed in a room with a constant temperature of 70°F, and 1 hour later its temperature is 52°F. Show that $t$ hours after the lemonade is placed in the room its temperature is approximated by $T = 70 - 30e^{-0.5t}$.

61. A rocket, fired upward from rest at time $t = 0$, has an initial mass of $m_0$ (including its fuel). Assuming that the fuel is consumed at a constant rate $k$, the mass $m$ of the rocket, while fuel is being burned, will be given by $m = m_0 - kt$.
It can be shown that if air resistance is neglected and the fuel gases are expelled at a constant speed $c$ relative to the rocket, then the velocity $v$ of the rocket will satisfy the equation

$$mdv/dt = ck - mg$$

where $g$ is the acceleration due to gravity.
(a) Find $v(t)$ keeping in mind that the mass $m$ is a function of $t$.
(b) Suppose that the fuel accounts for 80% of the initial mass of the rocket and that all of the fuel is consumed in 100 s. Find the velocity of the rocket in meters per second at the instant the fuel is exhausted. [Note: Take $g = 9.8$ m/s$^2$ and $c = 2500$ m/s.]

62. A bullet of mass $m$, fired straight up with an initial velocity of $v_0$, is slowed by the force of gravity and a drag force of air resistance $kv^2$, where $k$ is a positive constant. As the bullet moves upward, its velocity $v$ satisfies the equation

$$m dv/dt = -(kv^2 + mg)$$

where $g$ is the constant acceleration due to gravity.
(a) Show that if $x = x(t)$ is the height of the bullet above the barrel opening at time $t$, then

$$mv dv/dx = -(kv^2 + mg)$$

(b) Express $x$ in terms of $v$ given that $x = 0$ when $v = v_0$.
(c) Assuming that $v_0 = 988$ m/s, $g = 9.8$ m/s$^2$

$m = 3.56 \times 10^{-3}$ kg, $k = 7.3 \times 10^{-6}$ kg/m

use the result in part (b) to find out how high the bullet rises. [Hint: Find the velocity of the bullet at its highest point.]

63–64 Suppose that a tank containing a liquid is vented to the air at the top and has an outlet at the bottom through which the liquid can drain. It follows from Torricelli’s law in physics that if the outlet is opened at time $t = 0$, then at each instant the depth of the liquid $h(t)$ and the area $A(h)$ of the liquid’s surface are related by

$$A(h) dh/dt = -k\sqrt{h}$$

where $k$ is a positive constant that depends on such factors as the viscosity of the liquid and the cross-sectional area of the outlet. Use this result in these exercises, assuming that $h$ is in feet, $A(h)$ is in square feet, and $t$ is in seconds. ■

63. Suppose that the cylindrical tank in the accompanying figure is filled to a depth of 4 feet at time $t = 0$ and that the constant in Torricelli’s law is $k = 0.025$.
(a) Find $h(t)$.
(b) How many minutes will it take for the tank to drain completely?

64. Follow the directions of Exercise 63 for the cylindrical tank in the accompanying figure, assuming that the tank is filled to a depth of 4 feet at time $t = 0$ and that the constant in Torricelli’s law is $k = 0.025$. 

![Figure Ex-63](image1)
![Figure Ex-64](image2)
65. Suppose that a particle moving along the x-axis encounters a resisting force that results in an acceleration of \( a = \frac{dv}{dt} = -\frac{1}{32}v^2 \). If \( x = 0 \) cm and \( v = 128 \) cm/s at time \( t = 0 \), find the velocity \( v \) and position \( x \) as a function of \( t \) for \( t \geq 0 \).

66. Suppose that a particle moving along the x-axis encounters a resisting force that results in an acceleration of \( a = \frac{dv}{dt} = -0.02\sqrt{v} \). Given that \( x = 0 \) cm and \( v = 128 \) cm/s at time \( t = 0 \), find the velocity \( v \) and position \( x \) as a function of \( t \) for \( t \geq 0 \).

FOCUS ON CONCEPTS

67. Use implicit differentiation to prove that any differentiable function defined implicitly by Equation (4) will be a solution to (1).

68. Prove that a solution to the initial-value problem

\[
\frac{dy}{dx} = g(x), \quad y(x_0) = y_0
\]

is defined implicitly by the equation

\[
\int_{x_0}^{x} h(r) \, dr = \int_{y_0}^{y} g(s) \, ds
\]

69. Let \( L \) denote a tangent line at \((x, y)\) to a solution of Equation (1), and let \((x_1, y_1), (x_2, y_2)\) denote any two points on \( L \). Prove that Equation (2) is satisfied by \( \Delta y = y_2 - y_1 \) and \( \Delta x = x_2 - x_1 \).

70. Writing A student objects to the method of separation of variables because it often produces an equation in \( x \) and \( y \) instead of an explicit function \( y = f(x) \). Discuss the pros and cons of this student’s position.

71. Writing A student objects to Step 2 in the method of separation of variables because one side of the equation is integrated with respect to \( x \) while the other side is integrated with respect to \( y \). Answer this student’s objection. [Hint: Recall the method of integration by substitution.]

✔ QUICK CHECK ANSWERS 8.2

1. Step 1: \( h(y) \, dy = g(x) \, dx \); Step 2: \( \int h(y) \, dy = \int g(x) \, dx \); Step 3: \( H(y) = G(x) + C \)

2. (a) \( ky \) (b) \( \frac{\ln 2}{k} \) (c) \( y_0 e^{kt} \)

3. (a) \(-ky\) (b) \( \frac{\ln 2}{k} \) (c) \( y_0 e^{-kt} \)

4. \( y = \sqrt{1 - x^2} \)

8.3 SLOPE FIELDS; EULER’S METHOD

In this section we will reexamine the concept of a slope field and we will discuss a method for approximating solutions of first-order equations numerically. Numerical approximations are important in cases where the differential equation cannot be solved exactly.

FUNCTIONS OF TWO VARIABLES

We will be concerned here with first-order equations that are expressed with the derivative by itself on one side of the equation. For example,

\[
y' = x^3 \quad \text{and} \quad y' = \sin(xy)
\]

The first of these equations involves only \( x \) on the right side, so it has the form \( y' = f(x) \). However, the second equation involves both \( x \) and \( y \) on the right side, so it has the form \( y' = f(x, y) \), where the symbol \( f(x, y) \) stands for a function of the two variables \( x \) and \( y \). Later in the text we will study functions of two variables in more depth, but for now it will suffice to think of \( f(x, y) \) as a formula that produces a unique output when values of \( x \) and \( y \) are given as inputs. For example, if

\[
f(x, y) = x^2 + 3y
\]
and if the inputs are $x = 2$ and $y = -4$, then the output is

$$f(2, -4) = 2^2 + 3(-4) = 4 - 12 = -8$$

### SLOPE FIELDS

In Section 5.2 we introduced the concept of a slope field in the context of differential equations of the form $y' = f(x)$; the same principles apply to differential equations of the form $y' = f(x, y)$.

To see why this is so, let us review the basic idea. If we interpret $y'$ as the slope of a tangent line, then the differential equation states that at each point $(x, y)$ on an integral curve, the slope of the tangent line is equal to the value of $f$ at that point (Figure 8.3.1). For example, suppose that $f(x, y) = y - x$, in which case we have the differential equation

$$y' = y - x \quad (1)$$

A geometric description of the set of integral curves can be obtained by choosing a rectangular grid of points in the $xy$-plane, calculating the slopes of the tangent lines to the integral curves at the gridpoints, and drawing small segments of the tangent lines through those points. The resulting picture is called a slope field or a direction field for the differential equation because it shows the “slope” or “direction” of the integral curves at the gridpoints. The more gridpoints that are used, the better the description of the integral curves. For example, Figure 8.3.2 shows two slope fields for (1)—the first was obtained by hand calculation using the 49 gridpoints shown in the accompanying table, and the second, which gives a clearer picture of the integral curves, was obtained using 625 gridpoints and a CAS.

It so happens that Equation (1) can be solved exactly using a method we will introduce in Section 8.4. We leave it for you to confirm that the general solution of this equation is

$$y = x + 1 + Ce^t \quad (2)$$

Figure 8.3.3 shows some of the integral curves superimposed on the slope field. Note that it was not necessary to have the general solution to construct the slope field. Indeed, slope fields are important precisely because they can be constructed in cases where the differential equation cannot be solved exactly.
8.3 Slope Fields; Euler’s Method

Consider an initial-value problem of the form

$$y' = f(x, y), \quad y(x_0) = y_0$$

The slope field for the differential equation $y' = f(x, y)$ gives us a way to visualize the solution of the initial-value problem, since the graph of the solution is the integral curve that passes through the point $(x_0, y_0)$. The slope field will also help us to develop a method for approximating the solution to the initial-value problem numerically.

We will not attempt to approximate $y(x)$ for all values of $x$; rather, we will choose some small increment $\Delta x$ and focus on approximating the values of $y(x)$ at a succession of $x$-values spaced $\Delta x$ units apart, starting from $x_0$. We will denote these $x$-values by

$$x_1 = x_0 + \Delta x, \quad x_2 = x_1 + \Delta x, \quad x_3 = x_2 + \Delta x, \quad x_4 = x_3 + \Delta x, \ldots$$

and we will denote the approximations of $y(x)$ at these points by

$$y_1 \approx y(x_1), \quad y_2 \approx y(x_2), \quad y_3 \approx y(x_3), \quad y_4 \approx y(x_4), \ldots$$

The technique that we will describe for obtaining these approximations is called Euler’s Method. Although there are better approximation methods available, many of them use Euler’s Method as a starting point, so the underlying concepts are important to understand.

The basic idea behind Euler’s Method is to start at the known initial point $(x_0, y_0)$ and draw a line segment in the direction determined by the slope field until we reach the point $(x_1, y_1)$ with $x$-coordinate $x_1 = x_0 + \Delta x$ (Figure 8.3.4). If $\Delta x$ is small, then it is reasonable to expect that this line segment will not deviate much from the integral curve $y = y(x)$, and thus $y_1$ should closely approximate $y(x_1)$. To obtain the subsequent approximations, we repeat the process using the slope field as a guide at each step. Starting at the endpoint $(x_1, y_1)$, we draw a line segment determined by the slope field until we reach the point $(x_2, y_2)$ with $x$-coordinate $x_2 = x_1 + \Delta x$, and from that point we draw a line segment determined by the slope field to the point $(x_3, y_3)$ with $x$-coordinate $x_3 = x_2 + \Delta x$, and so forth. As indicated in Figure 8.3.4, this procedure produces a polygonal path that tends to follow the integral curve closely, so it is reasonable to expect that the $y$-values $y_2, y_3, y_4, \ldots$ will closely approximate $y(x_2), y(x_3), y(x_4), \ldots$.

To explain how the approximations $y_1, y_2, y_3, \ldots$ can be computed, let us focus on a typical line segment. As indicated in Figure 8.3.5, assume that we have found the point $(x_n, y_n)$, and we are trying to determine the next point $(x_{n+1}, y_{n+1})$, where $x_{n+1} = x_n + \Delta x$. Since the slope of the line segment joining the points is determined by the slope field at the starting point, the slope is $f(x_n, y_n)$, and hence

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{y_{n+1} - y_n}{\Delta x} = f(x_n, y_n)$$

which we can rewrite as

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x$$

This formula, which is the heart of Euler’s Method, tells us how to use each approximation to compute the next approximation.
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Euler’s Method

To approximate the solution of the initial-value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

proceed as follows:

**Step 1.** Choose a nonzero number \( \Delta x \) to serve as an increment or step size along the \( x \)-axis, and let

\[
\begin{align*}
  x_1 &= x_0 + \Delta x, \\
  x_2 &= x_1 + \Delta x, \\
  x_3 &= x_2 + \Delta x, & \ldots
\end{align*}
\]

**Step 2.** Compute successively

\[
\begin{align*}
  y_1 &= y_0 + f(x_0, y_0) \Delta x \\
  y_2 &= y_1 + f(x_1, y_1) \Delta x \\
  y_3 &= y_2 + f(x_2, y_2) \Delta x \\
  \vdots \\
  y_{n+1} &= y_n + f(x_n, y_n) \Delta x
\end{align*}
\]

The numbers \( y_1, y_2, y_3, \ldots \) in these equations are the approximations of \( y(x_1), y(x_2), y(x_3), \ldots \).

**Example 1** Use Euler’s Method with a step size of 0.1 to make a table of approximate values of the solution of the initial-value problem

\[ y' = y - x, \quad y(0) = 2 \] (3)

over the interval \( 0 \leq x \leq 1 \).

**Solution.** In this problem we have \( f(x, y) = y - x, x_0 = 0 \), and \( y_0 = 2 \). Moreover, since the step size is 0.1, the \( x \)-values at which the approximate values will be obtained are

\[
\begin{align*}
  x_1 &= 0.1, & x_2 &= 0.2, & x_3 &= 0.3, & \ldots & x_9 &= 0.9, & x_{10} &= 1
\end{align*}
\]

The first three approximations are

\[
\begin{align*}
  y_1 &= y_0 + f(x_0, y_0) \Delta x = 2 + (2 - 0)(0.1) = 2.2 \\
  y_2 &= y_1 + f(x_1, y_1) \Delta x = 2.2 + (2.2 - 0.1)(0.1) = 2.41 \\
  y_3 &= y_2 + f(x_2, y_2) \Delta x = 2.41 + (2.41 - 0.2)(0.1) = 2.631
\end{align*}
\]

Here is a way of organizing all 10 approximations rounded to five decimal places:

**Euler’s Method for \( y' = y - x, y(0) = 2 \) with \( \Delta x = 0.1 \)**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
<th>( f(x_n, y_n) \Delta x )</th>
<th>( y_{n+1} = y_n + f(x_n, y_n) \Delta x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2.00000</td>
<td>0.200000</td>
<td>2.200000</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>2.20000</td>
<td>0.210000</td>
<td>2.410000</td>
</tr>
<tr>
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<td>0.2</td>
<td>2.41000</td>
<td>0.221000</td>
<td>2.631000</td>
</tr>
<tr>
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<td>0.3</td>
<td>2.63100</td>
<td>0.233100</td>
<td>2.864100</td>
</tr>
<tr>
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<td>0.4</td>
<td>2.86410</td>
<td>0.246410</td>
<td>3.110514</td>
</tr>
<tr>
<td>5</td>
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<td>3.11051</td>
<td>0.261051</td>
<td>3.371564</td>
</tr>
<tr>
<td>6</td>
<td>0.6</td>
<td>3.37156</td>
<td>0.277160</td>
<td>3.648721</td>
</tr>
<tr>
<td>7</td>
<td>0.7</td>
<td>3.64872</td>
<td>0.294870</td>
<td>3.943590</td>
</tr>
<tr>
<td>8</td>
<td>0.8</td>
<td>3.94359</td>
<td>0.314360</td>
<td>4.257950</td>
</tr>
<tr>
<td>9</td>
<td>0.9</td>
<td>4.25795</td>
<td>0.335790</td>
<td>4.593740</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>4.59374</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
Observe that each entry in the last column becomes the next entry in the third column. This is reminiscent of Newton’s Method in which each successive approximation is used to find the next.

### ACCURACY OF EULER’S METHOD

It follows from (3) and the initial condition \( y(0) = 2 \) that the exact solution of the initial-value problem in Example 1 is

\[
y = x + 1 + e^x
\]

Thus, in this case we can compare the approximate values of \( y(x) \) produced by Euler’s Method with decimal approximations of the exact values (Table 8.3.1). In Table 8.3.1 the absolute error is calculated as

\[
\text{absolute error} = |\text{exact value} - \text{approximation}|
\]

and the percentage error as

\[
\text{percentage error} = \left( \frac{|\text{exact value} - \text{approximation}|}{\text{exact value}} \right) \times 100\%
\]

Table 8.3.1

<table>
<thead>
<tr>
<th>( x )</th>
<th>\text{EXACT SOLUTION}</th>
<th>\text{EULER APPROXIMATION}</th>
<th>\text{ABSOLUTE ERROR}</th>
<th>\text{PERCENTAGE ERROR}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.00000</td>
<td>2.00000</td>
<td>0.00000</td>
<td>0.00</td>
</tr>
<tr>
<td>0.1</td>
<td>2.20517</td>
<td>2.20000</td>
<td>0.00517</td>
<td>0.23</td>
</tr>
<tr>
<td>0.2</td>
<td>2.42140</td>
<td>2.41000</td>
<td>0.01140</td>
<td>0.47</td>
</tr>
<tr>
<td>0.3</td>
<td>2.64986</td>
<td>2.63100</td>
<td>0.01886</td>
<td>0.71</td>
</tr>
<tr>
<td>0.4</td>
<td>2.89182</td>
<td>2.86410</td>
<td>0.02772</td>
<td>0.96</td>
</tr>
<tr>
<td>0.5</td>
<td>3.14872</td>
<td>3.11051</td>
<td>0.03821</td>
<td>1.21</td>
</tr>
<tr>
<td>0.6</td>
<td>3.42212</td>
<td>3.37156</td>
<td>0.05056</td>
<td>1.48</td>
</tr>
<tr>
<td>0.7</td>
<td>3.71375</td>
<td>3.64872</td>
<td>0.06503</td>
<td>1.75</td>
</tr>
<tr>
<td>0.8</td>
<td>4.02554</td>
<td>3.94359</td>
<td>0.08195</td>
<td>2.04</td>
</tr>
<tr>
<td>0.9</td>
<td>4.35960</td>
<td>4.25795</td>
<td>0.10165</td>
<td>2.33</td>
</tr>
<tr>
<td>1.0</td>
<td>4.71828</td>
<td>4.59374</td>
<td>0.12454</td>
<td>2.64</td>
</tr>
</tbody>
</table>

Notice that the absolute error tends to increase as \( x \) moves away from \( x_0 \).

### QUICK CHECK EXERCISES 8.3

(See page 586 for answers.)

1. Match each differential equation with its slope field.
   (a) \( y' = 2xy \) ____________
   (b) \( y' = e^{-y} \) ____________
   (c) \( y' = y \) ____________
   (d) \( y' = 2xy \) ____________

   ![Figure Ex-1](https://via.placeholder.com/150)

2. The slope field for \( y' = y/x \) at the 16 gridpoints \((x, y)\), where \( x = -2, -1, 1, 2 \) and \( y = -2, -1, 1, 2 \) is shown in
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the accompanying figure. Use this slope field and geometric reasoning to find the integral curve that passes through the point (1, 2).

3. When using Euler’s Method on the initial-value problem $y' = f(x, y)$, $y(x_0) = y_0$, we obtain $y_{n+1}$ from $y_n, x_n$, and $\Delta x$ by means of the formula $y_{n+1} =$ ________.

4. Consider the initial-value problem $y' = y$, $y(0) = 1$.
   (a) Use Euler’s Method with two steps to approximate $y(1)$.
   (b) What is the exact value of $y(1)$?

EXERCISE SET 8.3

1. Sketch the slope field for $y' = xy/4$ at the 25 gridpoints $(x, y)$, where $x = -2, -1, \ldots, 2$ and $y = -2, -1, \ldots, 2$.

2. Sketch the slope field for $y' + y = 2$ at the 25 gridpoints $(x, y)$, where $x = 0, 1, \ldots, 4$ and $y = 0, 1, \ldots, 4$.

3. A slope field for the differential equation $y' = 1 - y$ is shown in the accompanying figure. In each part, sketch the graph of the solution that satisfies the initial condition.
   (a) $y(0) = -1$  (b) $y(0) = 1$  (c) $y(0) = 2$

4. Solve the initial-value problems in Exercise 3, and use a graphing utility to confirm that the integral curves for these solutions are consistent with the sketches you obtained from the slope field.

FOCUS ON CONCEPTS

5. Use the slope field in Exercise 3 to make a conjecture about the behavior of the solutions of $y' = 1 - y$ as $x \to +\infty$, and confirm your conjecture by examining the general solution of the equation.

6. In parts (a)–(f), match the differential equation with the slope field, and explain your reasoning.
   (a) $y' = 1/x$  (b) $y' = 1/y$
   (c) $y' = e^{-x^2}$  (d) $y' = y^2 - 1$
   (e) $y' = \frac{x + y}{x - y}$  (f) $y' = (\sin x)(\sin y)$
7–10 Use Euler’s Method with the given step size $\Delta x$ or $\Delta t$ to approximate the solution of the initial-value problem over the stated interval. Present your answer as a table and as a graph.

7. $dy/dx = \sqrt{3}, \; y(0) = 1, \; 0 \leq x \leq 4, \; \Delta x = 0.5$
8. $dy/dx = x - y^2, \; y(0) = 1, \; 0 \leq x \leq 2, \; \Delta x = 0.25$
9. $dy/dt = \cos y, \; y(0) = 1, \; 0 \leq t \leq 2, \; \Delta t = 0.5$
10. $dy/dt = e^{-t}, \; y(0) = 0, \; 0 \leq t \leq 1, \; \Delta t = 0.1$

11. Consider the initial-value problem

\[ y' = \sin \pi t, \; y(0) = 0 \]

Use Euler’s Method with five steps to approximate $y(1)$.

12–15 True–False Determine whether the statement is true or false. Explain your answer.

12. If the graph of $y = f(x)$ is an integral curve for a slope field, then so is any vertical translation of this graph.
13. Every integral curve for the slope field $dy/dx = e^{xy}$ is the graph of an increasing function of $x$.
14. Every integral curve for the slope field $dy/dx = e^y$ is concave up.
15. If $p(y)$ is a cubic polynomial in $y$, then the slope field $dy/dx = p(y)$ has an integral curve that is a horizontal line.

**FOCUS ON CONCEPTS**

16. (a) Show that the solution of the initial-value problem

\[ y' = e^{-x^2}, \; y(0) = 0 \]

is

\[ y(x) = \int_0^x e^{-t^2} \, dt \]

(b) Use Euler’s Method with $\Delta x = 0.05$ to approximate the value of

\[ y(1) = \int_0^1 e^{-t^2} \, dt \]

and compare the answer to that produced by a calculating utility with a numerical integration capability.

17. The accompanying figure shows a slope field for the differential equation $y' = -x/y$.

(a) Use the slope field to estimate $y(\frac{1}{2})$ for the solution that satisfies the given initial condition $y(0) = 1$.
(b) Compare your estimate to the exact value of $y(\frac{1}{2})$.

18. Refer to slope field II in Quick Check Exercise 1.
(a) Does the slope field appear to have a horizontal line as an integral curve?
(b) Use the differential equation for the slope field to verify your answer to part (a).

19. Refer to the slope field in Exercise 3 and consider the integral curve through $(0, -1)$.
(a) Use the slope field to estimate where the integral curve intersects the $x$-axis.
(b) Compare your estimate in part (a) with the exact value of the $x$-intercept for the integral curve.

20. Consider the initial-value problem

\[ \frac{dy}{dx} = \sqrt{x}, \; y(0) = 1 \]

(a) Use Euler’s Method with step sizes of $\Delta x = 0.2$, $0.1$, and $0.05$ to obtain three approximations of $y(1)$.
(b) Find $y(1)$ exactly.

21. A slope field of the form $y' = f(y)$ is said to be **autonomous**.

(a) Explain why the tangent segments along any horizontal line will be parallel for an autonomous slope field.
(b) The word autonomous means “independent.” In what sense is an autonomous slope field independent?
(c) Suppose that $G(y)$ is an antiderivative of $1/|f(y)|$ and that $C$ is a constant. Explain why any differentiable function defined implicitly by $G(y) - x = C$ will be a solution to the equation $y' = f(y)$.

22. (a) Solve the equation $y' = \sqrt{y}$ and show that every nonconstant solution has a graph that is everywhere concave up.
(b) Explain how the conclusion in part (a) may be obtained directly from the equation $y' = \sqrt{y}$ without solving.

23. (a) Find a slope field whose integral curve through $(1, 1)$ satisfies $xy^2 - x^2 y = 0$ by differentiating this equation implicitly.
(b) Prove that if $y(x)$ is any integral curve of the slope field in part (a), then $x[y(x)]^3 - x^2 y(x)$ will be a constant function.
(c) Find an equation that implicitly defines the integral curve through $(−1, −1)$ of the slope field in part (a).

24. (a) Find a slope field whose integral curve through $(0, 0)$ satisfies $xe^y + ye^x = 0$ by differentiating this equation implicitly.
(b) Prove that if $y(x)$ is any integral curve of the slope field in part (a), then $xe^{y(x)} + y(x)e^x$ will be a constant function.
(c) Find an equation that implicitly defines the integral curve through $(1, 1)$ of the slope field in part (a).
25. Consider the initial-value problem \( y' = y \), \( y(0) = 1 \), and let \( y_n \) denote the approximation of \( y(1) \) using Euler’s Method with \( n \) steps.
   (a) What would you conjecture is the exact value of \( \lim_{n \to +\infty} y_n \)? Explain your reasoning.
   (b) Find an explicit formula for \( y_n \) and use it to verify your conjecture in part (a).

26. Writing Explain the connection between Euler’s Method and the local linear approximation discussed in Section 3.5.

27. Writing Given a slope field, what features of an integral curve might be discussed from the slope field? Apply your ideas to the slope field in Exercise 3.

**QUICK CHECK ANSWERS 8.3**

1. (a) IV (b) III (c) I (d) II  
   2. \( y = 2x, x > 0 \)  
   3. \( y_n + f(x_n, y_n)\Delta x \)  
   4. (a) 2.25 (b) \( e \)

### 8.4 FIRST-ORDER DIFFERENTIAL EQUATIONS AND APPLICATIONS

In this section we will discuss a general method that can be used to solve a large class of first-order differential equations. We will use this method to solve differential equations related to the problems of mixing liquids and free fall retarded by air resistance.

#### FIRST-ORDER LINEAR EQUATIONS

The simplest first-order equations are those that can be written in the form
\[
\frac{dy}{dx} = q(x)
\]  
(1)

Such equations can often be solved by integration. For example, if
\[
\frac{dy}{dx} = x^3
\]  
(2)

then
\[
y = \int x^3 \, dx = \frac{x^4}{4} + C
\]
is the general solution of (2) on the interval \((-\infty, +\infty)\). More generally, a first-order differential equation is called \textit{linear} if it is expressible in the form
\[
\frac{dy}{dx} + p(x)y = q(x)
\]  
(3)

Equation (1) is the special case of (3) that results when the function \( p(x) \) is identically 0. Some other examples of first-order linear differential equations are
\[
\frac{dy}{dx} + x^2 y = e^x, \quad \frac{dy}{dx} + (\sin x)y + x^3 = 0, \quad \frac{dy}{dx} + 5y = 2
\]

\( p(x) = x^2, q(x) = e^x \) \quad \( p(x) = \sin x, q(x) = -x^3 \) \quad \( p(x) = 5, q(x) = 2 \)

We will assume that the functions \( p(x) \) and \( q(x) \) in (3) are continuous on a common interval, and we will look for a general solution that is valid on that interval. One method for doing this is based on the observation that if we define \( \mu = \mu(x) \) by
\[
\mu = e^{\int p(x) \, dx}
\]  
(4)
then
\[ \frac{d\mu}{dx} = e^{\int p(x)\,dx} \frac{d}{dx} \int p(x)\,dx = \mu p(x) \]

Thus,
\[ \frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \mu \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu p(x)y \]

If (3) is multiplied through by \( \mu \), it becomes
\[ \mu \frac{dy}{dx} + \mu p(x)y = \mu q(x) \]

Combining this with (5) we have
\[ \frac{d}{dx}(\mu y) = \mu q(x) \]

This equation can be solved for \( y \) by integrating both sides with respect to \( x \) and then dividing through by \( \mu \) to obtain
\[ y = \frac{1}{\mu} \int \mu q(x) \, dx \]

which is a general solution of (3) on the interval. The function \( \mu \) in (4) is called an **integrating factor** for (3), and this method for finding a general solution of (3) is called the **method of integrating factors**. Although one could simply memorize Formula (7), we recommend solving first-order linear equations by actually carrying out the steps used to derive this formula:

**The Method of Integrating Factors**

**Step 1.** Calculate the integrating factor
\[ \mu = e^{\int p(x)\,dx} \]

Since any \( \mu \) will suffice, we can take the constant of integration to be zero in this step.

**Step 2.** Multiply both sides of (3) by \( \mu \) and express the result as
\[ \frac{d}{dx}(\mu y) = \mu q(x) \]

**Step 3.** Integrate both sides of the equation obtained in Step 2 and then solve for \( y \). Be sure to include a constant of integration in this step.

**Example 1** Solve the differential equation
\[ \frac{dy}{dx} - y = e^{2x} \]

**Solution.** Comparing the given equation to (3), we see that we have a first-order linear equation with \( p(x) = -1 \) and \( q(x) = e^{2x} \). These coefficients are continuous on the interval \((-\infty, +\infty)\), so the method of integrating factors will produce a general solution on this interval. The first step is to compute the integrating factor. This yields
\[ \mu = e^{\int p(x)\,dx} = e^{\int (-1)\,dx} = e^{-x} \]
Next we multiply both sides of the given equation by \( \mu \) to obtain

\[
e^{-x} \frac{dy}{dx} - xe^{-x} y = e^{-x} e^{2x}
\]

which we can rewrite as

\[
\frac{d}{dx} [e^{-x} y] = e^x
\]

Integrating both sides of this equation with respect to \( x \) we obtain

\[e^{-x} y = e^x + C\]

Finally, solving for \( y \) yields the general solution

\[y = e^{2x} + Ce^x\]

A differential equation of the form

\[P(x) \frac{dy}{dx} + Q(x)y = R(x)\]

can be solved by dividing through by \( P(x) \) to put the equation in the form of (3) and then applying the method of integrating factors. However, the resulting solution will only be valid on intervals where \( p(x) = Q(x)/P(x) \) and \( q(x) = R(x)/P(x) \) are both continuous.

**Example 2** Solve the initial-value problem

\[
\frac{dy}{dx} - y = x, \quad y(1) = 2
\]

**Solution.** This differential equation can be written in the form of (3) by dividing through by \( x \). This yields

\[
\frac{dy}{dx} - \frac{1}{x} y = 1
\]

where \( q(x) = 1 \) is continuous on \((-\infty, +\infty)\) and \( p(x) = -1/x \) is continuous on \((-\infty, 0)\) and \((0, +\infty)\). Since we need \( p(x) \) and \( q(x) \) to be continuous on a common interval, and since our initial condition requires a solution for \( x = 1 \), we will find a general solution of (8) on the interval \((0, +\infty)\). On this interval we have \(|x| = x\), so that

\[
\int p(x) \, dx = -\int \frac{1}{x} \, dx = -\ln |x| = -\ln x
\]

Thus, an integrating factor that will produce a general solution on the interval \((0, +\infty)\) is

\[
\mu = e^{\int p(x) \, dx} = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}
\]

Multiplying both sides of Equation (8) by this integrating factor yields

\[
\frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = \frac{1}{x}
\]

or

\[
\frac{d}{dx} \left[ \frac{y}{x} \right] = \frac{1}{x}
\]
Therefore, on the interval $(0, +\infty)$,
\[
\frac{1}{x} y = \int \frac{1}{x} \, dx = \ln x + C
\]
from which it follows that
\[
y = x \ln x + Cx
\]
(9)
The initial condition $y(1) = 2$ requires that $y = 2$ if $x = 1$. Substituting these values into (9) and solving for $C$ yields $C = 2$ (verify), so the solution of the initial-value problem is
\[
y = x \ln x + 2x
\]
We conclude this section with some applications of first-order differential equations.

### MIXING PROBLEMS

In a typical mixing problem, a tank is filled to a specified level with a solution that contains a known amount of some soluble substance (say salt). The thoroughly stirred solution is allowed to drain from the tank at a known rate, and at the same time a solution with a known concentration of the soluble substance is added to the tank at a known rate that may or may not differ from the draining rate. As time progresses, the amount of the soluble substance in the tank will generally change, and the usual mixing problem seeks to determine the amount of the substance in the tank at a specified time. This type of problem serves as a model for many kinds of problems: discharge and filtration of pollutants in a river, injection and absorption of medication in the bloodstream, and migrations of species into and out of an ecological system, for example.

#### Example 3

At time $t = 0$, a tank contains 4 lb of salt dissolved in 100 gal of water. Suppose that brine containing 2 lb of salt per gallon of brine is allowed to enter the tank at a rate of 5 gal/min and that the mixed solution is drained from the tank at the same rate (Figure 8.4.1). Find the amount of salt in the tank after 10 minutes.

**Solution.** Let $y(t)$ be the amount of salt (in pounds) after $t$ minutes. We are given that $y(0) = 4$, and we want to find $y(10)$. We will begin by finding a differential equation that is satisfied by $y(t)$. To do this, observe that $\frac{dy}{dt}$, which is the rate at which the amount of salt in the tank changes with time, can be expressed as
\[
\frac{dy}{dt} = \text{rate in} - \text{rate out}
\]
where rate in is the rate at which salt enters the tank and rate out is the rate at which salt leaves the tank. But the rate at which salt enters the tank is
\[
\text{rate in} = (2 \text{ lb/gal}) \cdot (5 \text{ gal/min}) = 10 \text{ lb/min}
\]
Since brine enters and drains from the tank at the same rate, the volume of brine in the tank stays constant at 100 gal. Thus, after $t$ minutes have elapsed, the tank contains $y(t)$ lb of salt per 100 gal of brine, and hence the rate at which salt leaves the tank at that instant is
\[
\text{rate out} = \left( \frac{y(t)}{100} \text{ lb/gal} \right) \cdot (5 \text{ gal/min}) = \frac{y(t)}{20} \text{ lb/min}
\]
Therefore, (10) can be written as
\[
\frac{dy}{dt} = 10 - \frac{y}{20} \quad \text{or} \quad \frac{dy}{dt} + \frac{y}{20} = 10
\]
which is a first-order linear differential equation satisfied by \( y(t) \). Since we are given that \( y(0) = 4 \), the function \( y(t) \) can be obtained by solving the initial-value problem

\[
\frac{dy}{dt} + \frac{y}{20} = 10, \quad y(0) = 4
\]

The integrating factor for the differential equation is

\[
\mu = e^{\int \left(\frac{1}{20}\right) dt} = e^{t/20}
\]

If we multiply the differential equation through by \( \mu \), then we obtain

\[
\frac{d}{dt}(e^{t/20} y) = 10 e^{t/20}
\]

\[
e^{t/20} y = \int 10 e^{t/20} dt = 200 e^{t/20} + C
\]

\[
y(t) = 200 + C e^{-t/20} \quad (11)
\]

The initial condition states that \( y = 4 \) when \( t = 0 \). Substituting these values into (11) and solving for \( C \) yields \( C = -196 \) (verify), so

\[
y(t) = 200 - 196 e^{-t/20} \quad (12)
\]

The graph of (12) is shown in Figure 8.4.2. At time \( t = 10 \) the amount of salt in the tank is

\[
y(10) = 200 - 196 e^{-0.5} \approx 81.1 \text{ lb}\]

Notice that it follows from (11) that

\[
\lim_{t \to +\infty} y(t) = 200
\]

for all values of \( C \), so regardless of the amount of salt that is present in the tank initially, the amount of salt in the tank will eventually stabilize at 200 lb. This can also be seen geometrically from the slope field for the differential equation shown in Figure 8.4.3. This slope field suggests the following: If the amount of salt present in the tank is greater than 200 lb initially, then the amount of salt will decrease steadily over time toward a limiting value of 200 lb; and if the amount of salt is less than 200 lb initially, then it will increase steadily toward a limiting value of 200 lb. The slope field also suggests that if the amount present initially is exactly 200 lb, then the amount of salt in the tank will stay constant at 200 lb. This can also be seen from (11), since \( C = 0 \) in this case (verify).

### A Model of Free-Fall Motion Retarded by Air Resistance

In Section 5.7 we considered the free-fall model of an object moving along a vertical axis near the surface of the Earth. It was assumed in that model that there is no air resistance and that the only force acting on the object is the Earth’s gravity. Our goal here is to find a model that takes air resistance into account. For this purpose we make the following assumptions:

- The object moves along a vertical \( s \)-axis whose origin is at the surface of the Earth and whose positive direction is up (Figure 5.7.7).
- At time \( t = 0 \) the height of the object is \( s_0 \) and the velocity is \( v_0 \).
- The only forces on the object are the force \( F_g = -mg \) of the Earth’s gravity acting down and the force \( F_R \) of air resistance acting opposite to the direction of motion. The force \( F_R \) is called the drag force.
In the case of free-fall motion retarded by air resistance, the net force acting on the object is
\[ F_G + F_R = -mg + F_R \]
and the acceleration is \( \frac{d^2s}{dt^2} \), so Newton’s Second Law of Motion [Equation (5) of Section 6.6] implies that
\[ -mg + F_R = m \frac{d^2s}{dt^2} \] \hspace{1cm} (13)

Experimentation has shown that the force \( F_R \) of air resistance depends on the shape of the object and its speed—the greater the speed, the greater the drag force. There are many possible models for air resistance, but one of the most basic assumes that the drag force \( F_R \) is proportional to the velocity of the object, that is,
\[ F_R = -cv \]
where \( c \) is a positive constant that depends on the object’s shape and properties of the air. (The minus sign ensures that the drag force is opposite to the direction of motion.) Substituting this in (13) and writing \( \frac{d^2s}{dt^2} \) as \( \frac{dv}{dt} \), we obtain
\[ -mg - cv = m \frac{dv}{dt} \]
Dividing by \( m \) and rearranging we obtain
\[ \frac{dv}{dt} + \frac{c}{m} v = -g \]
which is a first-order linear differential equation in the unknown function \( v = v(t) \) with \( p(t) = c/m \) and \( q(t) = -g \) [see (3)]. For a specific object, the coefficient \( c \) can be determined experimentally, so we will assume that \( m, g, \) and \( c \) are known constants. Thus, the velocity function \( v = v(t) \) can be obtained by solving the initial-value problem
\[ \frac{dv}{dt} + \frac{c}{m} v = -g, \quad v(0) = v_0 \] \hspace{1cm} (14)
Once the velocity function is found, the position function \( s = s(t) \) can be obtained by solving the initial-value problem
\[ \frac{ds}{dt} = v(t), \quad s(0) = s_0 \] \hspace{1cm} (15)
In Exercise 25 we will ask you to solve (14) and show that
\[ v(t) = e^{-ct/m} \left( v_0 + \frac{mg}{c} \right) - \frac{mg}{c} \] \hspace{1cm} (16)
Note that
\[ \lim_{t \to +\infty} v(t) = -\frac{mg}{c} \] \hspace{1cm} (17)
(verify). Thus, the speed \( |v(t)| \) does not increase indefinitely, as in free fall; rather, because of the air resistance, it approaches a finite limiting speed \( v_\tau \), given by
\[ v_\tau = \left| -\frac{mg}{c} \right| = \frac{mg}{c} \] \hspace{1cm} (18)
This is called the terminal speed of the object, and (17) is called its terminal velocity.

**Remark**
Intuition suggests that near the limiting velocity, the velocity \( v(t) \) changes very slowly; that is, \( dv/dt \approx 0 \). Thus, it should not be surprising that the limiting velocity can be obtained informally from (14) by setting \( dv/dt = 0 \) in the differential equation and solving for \( v \). This yields
\[ v = -\frac{mg}{c} \]
which agrees with (17).

\* Other common models assume that \( F_R = -cv^2 \) or, more generally, \( F_R = -cv^p \) for some value of \( p \).
✔ QUICK CHECK EXERCISES 8.4  (See page 594 for answers.)

1. Solve the first-order linear differential equation
   \[ \frac{dy}{dx} + p(x)y = q(x) \]
   by completing the following steps:
   
   Step 1. Calculate the integrating factor \( \mu = \) ________.
   
   Step 2. Multiply both sides of the equation by the integrating factor and express the result as
   \[ \frac{d}{dx} \left[ \text{_______} \right] = \] ________
   
   Step 3. Integrate both sides of the equation obtained in Step 2 and solve for \( y = \) ________.

2. An integrating factor for
   \[ \frac{dy}{dx} + y \cdot x = q(x) \]
   is ________

3. At time \( t = 0 \), a tank contains 30 oz of salt dissolved in 60 gal of water. Then brine containing 5 oz of salt per gallon of brine is allowed to enter the tank at a rate of 3 gal/min and the mixed solution is drained from the tank at the same rate. Give an initial-value problem satisfied by the amount of salt \( y(t) \) in the tank at time \( t \). Do not solve the problem.

EXERCISE SET 8.4  (Graphing Utility)

1–6 Solve the differential equation by the method of integrating factors.

1. \( \frac{dy}{dx} + 4y = e^{-3x} \)  
2. \( \frac{dy}{dx} + 2xy = x \)
3. \( y' + y = \cos(e^t) \)
4. \( 2\frac{dy}{dx} + 4y = 1 \)
5. \((x^2 + 1)\frac{dy}{dx} + xy = 0 \)  
6. \( \frac{dy}{dx} + y + \frac{1}{1 - e^x} = 0 \)

7–10 Solve the initial-value problem.

7. \( x \frac{dy}{dx} + y = x, \quad y(1) = 2 \)
8. \( x \frac{dy}{dx} - y = x^2, \quad y(1) = -1 \)
9. \( \frac{dy}{dx} - 2xy = 2x, \quad y(0) = 3 \)
10. \( \frac{dy}{dt} + y = 2, \quad y(0) = 1 \)

11–14 True–False Determine whether the statement is true or false. Explain your answer.

11. If \( y_1 \) and \( y_2 \) are two solutions to a first-order linear differential equation, then \( y = y_1 + y_2 \) is also a solution.
12. If the first-order linear differential equation
   \[ \frac{dy}{dx} + p(x)y = q(x) \]
   has a solution that is a constant function, then \( q(x) \) is a constant multiple of \( p(x) \).
13. In a mixing problem, we expect the concentration of the dissolved substance within the tank to approach a finite limit over time.
14. In our model for free-fall motion retarded by air resistance, the terminal velocity is proportional to the weight of the falling object.
15. A slope field for the differential equation \( y' = 2y - x \) is shown in the accompanying figure. In each part, sketch the graph of the solution that satisfies the initial condition.
   (a) \( y(1) = 1 \)  
   (b) \( y(0) = -1 \)  
   (c) \( y(-1) = 0 \)

16. Solve the initial-value problems in Exercise 15, and use a graphing utility to confirm that the integral curves for these solutions are consistent with the sketches you obtained from the slope field.

17. Use the slope field in Exercise 15 to make a conjecture about the effect of \( y_0 \) on the behavior of the solution of the initial-value problem \( y' = 2y - x, \quad y(0) = y_0 \) as \( x \to +\infty \), and check your conjecture by examining the solution of the initial-value problem.
18. Consider the slope field in Exercise 15.
   (a) Use Euler’s Method with \( \Delta x = 0.1 \) to estimate \( y\left(\frac{1}{2}\right) \) for the solution that satisfies the initial condition \( y(0) = 1 \).
(b) Would you conjecture your answer in part (a) to be greater than or less than the actual value of \( y(\frac{1}{2}) \)? Explain.

(c) Check your conjecture in part (b) by finding the exact value of \( y(\frac{1}{2}) \).

19. (a) Use Euler’s Method with a step size of \( \Delta x = 0.2 \) to approximate the solution of the initial-value problem
\[
y'(x) = x + y, \quad y(0) = 1
\]
over the interval \( 0 \leq x \leq 1 \).

(b) Solve the initial-value problem exactly, and calculate the error and the percentage error in each of the approximations in part (a).

(c) Sketch the exact solution and the approximate solution together.

20. It was stated at the end of Section 8.3 that reducing the step size in Euler’s Method by half reduces the error in each approximation by about half. Confirm that the error in \( y(1) \) is reduced by about half if a step size of \( \Delta x = 0.1 \) is used in Exercise 19.

21. At time \( t = 0 \), a tank contains 25 oz of salt dissolved in 50 gal of water. Then brine containing 4 oz of salt per gallon of brine is allowed to enter the tank at a rate of 2 gal/min and the mixed solution is drained from the tank at the same rate.

(a) How much salt is in the tank at an arbitrary time \( t \)?

(b) How much salt is in the tank after 25 min?

22. A tank initially contains 200 gal of pure water. Then at time \( t = 0 \) brine containing 4 oz of salt per gallon of brine is allowed to enter the tank at a rate of 20 gal/min and the mixed solution is drained from the tank at the same rate.

(a) How much salt is in the tank at an arbitrary time \( t \)?

(b) How much salt is in the tank after 30 min?

23. A tank with a 1000 gal capacity initially contains 500 gal of water that is polluted with 50 lb of particulate matter. At time \( t = 0 \), pure water is added at a rate of 20 gal/min and the mixed solution is drained off at a rate of 10 gal/min. How much particulate matter is in the tank when it reaches the point of overflowing?

24. The water in a polluted lake initially contains 1 lb of mercury salts per 100,000 gal of water. The lake is circular with diameter 30 m and uniform depth 3 m. Polluted water is pumped from the lake at a rate of 1000 gal/h and is replaced with fresh water at the same rate. Construct a table that shows the amount of mercury in the lake (in lb) at the end of each hour over a 12-hour period. Discuss any assumptions you made. [Note: Use 1 m³ = 264 gal.]

25. (a) Use the method of integrating factors to derive solution (16) to the initial-value problem (14). [Note: Keep in mind that \( c, m, \) and \( g \) are constants.]

(b) Show that (16) can be expressed in terms of the terminal speed (18) as
\[
v(t) = e^{-gt/v_0} (v_0 + v_t) - v_t
\]

(c) Show that if \( s(0) = s_0 \), then the position function of the object can be expressed as
\[
s(t) = s_0 - v_t t + \frac{v_t}{g} (v_0 + v_t) (1 - e^{-gt/v_0})
\]

26. Suppose a fully equipped skydiver weighing 240 lb has a terminal speed of 120 ft/s with a closed parachute and 24 ft/s with an open parachute. Suppose further that this skydiver is dropped from an airplane at an altitude of 10,000 ft, falls for 25 s with a closed parachute, and then falls the rest of the way with an open parachute.

(a) Assuming that the skydiver’s initial vertical velocity is zero, use Exercise 25 to find the skydiver’s vertical velocity and height at the time the parachute opens. [Note: Take \( g = 32 \text{ ft/s}^2 \).]

(b) Use a calculating utility to find a numerical solution for the total time that the skydiver is in the air.

27. The accompanying figure is a schematic diagram of a basic \( RL \) series electrical circuit that contains a power source with a time-dependent voltage of \( V(t) \) volts (V), a resistor with a constant resistance of \( R \) ohms (\( \Omega \)), and an inductor with a constant inductance of \( L \) henrys (H). If you don’t know anything about electrical circuits, don’t worry; all you need to know is that electrical theory states that a current of \( I(t) \) amperes (A) flows through the circuit where \( I(t) \) satisfies the differential equation
\[
L \frac{dI}{dt} + RI = V(t)
\]

(a) Find \( I(t) \) if \( R = 10 \Omega, L = 5 \text{H}, V \) is a constant 20 V, and \( I(0) = 0 \text{A} \).

(b) What happens to the current over a long period of time?

28. Find \( I(t) \) for the electrical circuit in Exercise 27 if \( R = 6 \Omega, L = 3 \text{H}, V(t) = 3 \sin t \text{V}, \) and \( I(0) = 15 \text{A} \).

29. (a) Prove that any function \( y = y(x) \) defined by Equation (7) will be a solution to (3).

(b) Consider the initial-value problem
\[
\frac{dy}{dx} + p(x)y = q(x), \quad y(x_0) = y_0
\]
where the functions \( p(x) \) and \( q(x) \) are both continuous on some open interval. Using the general solution for a first-order linear equation, prove that this initial-value problem has a unique solution on the interval.
30. (a) Prove that solutions need not be unique for nonlinear initial-value problems by finding two solutions to
\[ \frac{dy}{dx} = x, \quad y(0) = 0 \]
(b) Prove that solutions need not exist for nonlinear initial-value problems by showing that there is no solution for
\[ \frac{dy}{dx} = -x, \quad y(0) = 0 \]

31. **Writing** Explain why the quantity \( \mu \) in the Method of Integrating Factors is called an “integrating factor” and explain its role in this method.

32. **Writing** Suppose that a given first-order differential equation can be solved both by the method of integrating factors and by separation of variables. Discuss the advantages and disadvantages of each method.

**QUICK CHECK ANSWERS 8.4**

1. Step 1: \( e^{\int \mu(x) \, dx} \); Step 2: \( \mu y, \mu q(x) \); Step 3: \( \frac{1}{\mu} \int \mu q(x) \, dx \)

2. 2

3. \( \frac{dy}{dt} + \frac{y}{20} = 15, \ y(0) = 30 \)

**CHAPTER 8 REVIEW EXERCISES**

1. Classify the following first-order differential equations as separable, linear, both, or neither.
   - (a) \( \frac{dy}{dx} - 3y = \sin x \)
   - (b) \( \frac{dy}{dx} + xy = x \)
   - (c) \( \frac{dy}{dx} - x = 1 \)
   - (d) \( \frac{dy}{dx} + xy^2 = \sin(xy) \)

2. Which of the given differential equations are separable?
   - (a) \( \frac{dy}{dx} = f(x)g(y) \)
   - (b) \( \frac{dy}{dx} = \frac{f(x)}{g(y)} \)
   - (c) \( \frac{dy}{dx} = f(x) + g(y) \)
   - (d) \( \frac{dy}{dx} = \sqrt{f(x)g(y)} \)

3–5 Solve the differential equation by the method of separation of variables.
   - 3. \( \frac{dy}{dx} = (1 + y^2)x^2 \)
   - 4. \( 3 \tan y - \frac{dy}{dx} \sec x = 0 \)
   - 5. \( (1 + y^2)y' = e^x y \)

6–8 Solve the initial-value problem by the method of separation of variables.
   - 6. \( y' = 1 + y^2, \ y(0) = 1 \)
   - 7. \( y' = \frac{y^5}{x(1 + y^2)}, \ y(1) = 1 \)
   - 8. \( y' = 4y^2 \sec^2 x, \ y(\pi/8) = 1 \)
   - 9. Sketch the integral curve of \( y' = -2xy^2 \) that passes through the point \((0, 1)\).
   - 10. Sketch the integral curve of \( 2xy' = 1 \) that passes through the point \((0, 1)\) and the integral curve that passes through the point \((0, -1)\).
   - 11. Sketch the slope field for \( y' = xy/8 \) at the 25 gridpoints \((x, y)\), where \( x = 0, 1, \ldots, 4 \) and \( y = 0, 1, \ldots, 4 \).
   - 12. Solve the differential equation \( y' = xy/8 \), and find a family of integral curves for the slope field in Exercise 11.

13–14 Use Euler’s Method with the given step size \( \Delta x \) to approximate the solution of the initial-value problem over the stated interval. Present your answer as a table and as a graph.
   - 13. \( dy/dx = \sqrt{x}, \ y(0) = 1, \ 0 \leq x \leq 4, \ \Delta x = 0.5 \)
   - 14. \( dy/dx = \sin y, \ y(0) = 1, \ 0 \leq x \leq 2, \ \Delta x = 0.5 \)

15. Consider the initial-value problem
   \[ y' = \cos 2\pi t, \quad y(0) = 1 \]
   Use Euler’s Method with five steps to approximate \( y(1) \).

16. Cloth found in an Egyptian pyramid contains 78.5% of its original carbon-14. Estimate the age of the cloth.

17. In each part, find an exponential growth model \( y = ye^{kt} \) that satisfies the stated conditions.
   - (a) \( y_0 = 2 \); doubling time \( T = 5 \)
   - (b) \( y(0) = 5 \); growth rate 1.5%
   - (c) \( y(1) = 1 \); doubling time \( T = 100 \)
   - (d) \( y(1) = 1 \); doubling time \( T = 5 \)

18. Suppose that an initial population of 5000 bacteria grows exponentially at a rate of 1% per hour and that \( y = y(t) \) is the number of bacteria present after \( t \) hours.
   - (a) Find an initial-value problem whose solution is \( y(t) \).
   - (b) Find a formula for \( y(t) \).
   - (c) What is the doubling time for the population?
   - (d) How long does it take for the population of bacteria to reach 30,000?

19–20 Solve the differential equation by the method of integrating factors.
   - 19. \( \frac{dy}{dx} + 3y = e^{-2x} \)
   - 20. \( \frac{dy}{dx} + y - \frac{1}{1 + e^x} = 0 \)

21–23 Solve the initial-value problem by the method of integrating factors.
21. \( y' - xy = x, \ y(0) = 3 \)
22. \( xy' + 2y = 4x^2, \ y(1) = 2 \)
23. \( y' \cosh x + y \sinh x = \cosh^2 x, \ y(0) = 2 \)

\[ 20 \]

24. (a) Solve the initial-value problem
\[ y' - y = x \sin 3x, \quad y(0) = 1 \]
by the method of integrating factors, using a CAS to perform any difficult integrations.
(b) Use the CAS to solve the initial-value problem directly, and confirm that the answer is consistent with that obtained in part (a).
(c) Graph the solution.

25. A tank contains 1000 gal of fresh water. At time \( t = 0 \) min, brine containing 5 oz of salt per gallon of brine is poured into the tank at a rate of 10 gal/min, and the mixed solution is drained from the tank at the same rate. After 15 min that process is stopped and fresh water is poured into the tank at the rate of 5 gal/min, and the mixed solution is drained from the tank at the same rate. Find the amount of salt in the tank at time \( t = 30 \) min.

26. Suppose that a room containing 1200 ft\(^3\) of air is free of carbon monoxide. At time \( t = 0 \) cigarette smoke containing 4% carbon monoxide is introduced at the rate of 0.1 ft\(^3\)/min, and the well-circulated mixture is vented from the room at the same rate.
(a) Find a formula for the percentage of carbon monoxide in the room at time \( t \).
(b) Extended exposure to air containing 0.012% carbon monoxide is considered dangerous. How long will it take to reach this level?


---

**CHAPTER 8 MAKING CONNECTIONS**

1. Consider the first-order differential equation
\[
\frac{dy}{dx} + py = q
\]
where \( p \) and \( q \) are constants. If \( y = y(x) \) is a solution to this equation, define \( u = u(x) = q - py(x) \).
(a) Without solving the differential equation, show that \( u \) grows exponentially as a function of \( x \) if \( p < 0 \), and decays exponentially as a function of \( x \) if \( 0 < p \).
(b) Use the result of part (a) and Equations (13–14) of Section 8.2 to solve the initial-value problem
\[
\frac{dy}{dx} + 2y = 4, \quad y(0) = -1
\]

2. Consider a differential equation of the form
\[
\frac{dy}{dx} = f(ax + by + c)
\]
where \( f \) is a function of a single variable. If \( y = y(x) \) is a solution to this equation, define \( u = u(x) = ax + by(x) + c \).
(a) Find a separable differential equation that is satisfied by the function \( u \).
(b) Use your answer to part (a) to solve
\[
\frac{dy}{dx} = \frac{1}{x + y}
\]

3. A first-order differential equation is **homogeneous** if it can be written in the form
\[
\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{for } x \neq 0
\]
where \( f \) is a function of a single variable. If \( y = y(x) \) is a solution to a first-order homogeneous differential equation, define \( u = u(x) = y(x)/x \).
(a) Find a separable differential equation that is satisfied by the function \( u \).
(b) Use your answer to part (a) to solve
\[
\frac{dy}{dx} = \frac{x - y}{x + y}
\]

4. A first-order differential equation is called a **Bernoulli equation** if it can be written in the form
\[
\frac{dy}{dx} + p(x)y = q(x)y^n \quad \text{for } n \neq 0, 1
\]
If \( y = y(x) \) is a solution to a Bernoulli equation, define \( u = u(x) = [y(x)]^{1-n} \).
(a) Find a first-order linear differential equation that is satisfied by \( u \).
(b) Use your answer to part (a) to solve the initial-value problem
\[
x \frac{dy}{dx} - y = -2xy^2, \quad y(1) = \frac{1}{2}
\]
Perspective creates the illusion that the sequence of railroad ties continues indefinitely but converges toward a single point infinitely far away.

In this chapter we will be concerned with infinite series, which are sums that involve infinitely many terms. Infinite series play a fundamental role in both mathematics and science—they are used, for example, to approximate trigonometric functions and logarithms, to solve differential equations, to evaluate difficult integrals, to create new functions, and to construct mathematical models of physical laws. Since it is impossible to add up infinitely many numbers directly, one goal will be to define exactly what we mean by the sum of an infinite series. However, unlike finite sums, it turns out that not all infinite series actually have a sum, so we will need to develop tools for determining which infinite series have sums and which do not. Once the basic ideas have been developed we will begin to apply our work; we will show how infinite series are used to evaluate such quantities as \( \ln 2 \), \( e \), \( \sin 3^\circ \), and \( \pi \), how they are used to create functions, and finally, how they are used to model physical laws.

### 9.1 SEQUENCES

In everyday language, the term “sequence” means a succession of things in a definite order—chronological order, size order, or logical order, for example. In mathematics, the term “sequence” is commonly used to denote a succession of numbers whose order is determined by a rule or a function. In this section, we will develop some of the basic ideas concerning sequences of numbers.

**DEFINITION OF A SEQUENCE**

Stated informally, an infinite sequence, or more simply a sequence, is an unending succession of numbers, called terms. It is understood that the terms have a definite order; that is, there is a first term \( a_1 \), a second term \( a_2 \), a third term \( a_3 \), a fourth term \( a_4 \), and so forth. Such a sequence would typically be written as

\[
a_1, a_2, a_3, a_4, \ldots
\]

where the dots are used to indicate that the sequence continues indefinitely. Some specific examples are

\[
1, 2, 3, 4, \ldots \quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots
\]

\[
2, 4, 6, 8, \ldots \quad 1, -1, 1, -1, \ldots
\]

Each of these sequences has a definite pattern that makes it easy to generate additional terms if we assume that those terms follow the same pattern as the displayed terms. However,
9.1 Sequences

Such patterns can be deceiving, so it is better to have a rule or formula for generating the terms. One way of doing this is to look for a function that relates each term in the sequence to its term number. For example, in the sequence

\[ 2, 4, 6, 8, \ldots \]

each term is twice the term number; that is, the \( n \)th term in the sequence is given by the formula \( 2n \). We denote this by writing the sequence as

\[ 2, 4, 6, 8, \ldots, 2n, \ldots \]

We call the function \( f(n) = 2n \) the general term of this sequence. Now, if we want to know a specific term in the sequence, we need only substitute its term number in the formula for the general term. For example, the 37th term in the sequence is \( 2 \cdot 37 = 74 \).

**Example 1**

In each part, find the general term of the sequence.

(a) \( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \)

(b) \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \)

(c) \( \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots \)

(d) \( 1, 3, 5, 7, \ldots \)

**Solution (a).** In Table 9.1.1, the four known terms have been placed below their term numbers, from which we see that the numerator is the same as the term number and the denominator is one greater than the term number. This suggests that the \( n \)th term has numerator \( n \) and denominator \( n + 1 \), as indicated in the table. Thus, the sequence can be expressed as

\[ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots \]

**Solution (b).** In Table 9.1.2, the denominators of the four known terms have been expressed as powers of 2 and the first four terms have been placed below their term numbers, from which we see that the exponent in the denominator is the same as the term number. This suggests that the denominator of the \( n \)th term is \( 2^n \), as indicated in the table. Thus, the sequence can be expressed as

\[ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots \]

**Solution (c).** This sequence is identical to that in part (a), except for the alternating signs. Thus, the \( n \)th term in the sequence can be obtained by multiplying the \( n \)th term in part (a) by \((-1)^{n+1}\). This factor produces the correct alternating signs, since its successive values, starting with \( n = 1 \), are 1, \(-1\), 1, \(-1\), \ldots. Thus, the sequence can be written as

\[ \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1} \frac{n}{n+1}, \ldots \]

**Solution (d).** In Table 9.1.3, the four known terms have been placed below their term numbers, from which we see that each term is one less than twice its term number. This suggests that the \( n \)th term in the sequence is \( 2n - 1 \), as indicated in the table. Thus, the sequence can be expressed as

\[ 1, 3, 5, 7, \ldots, 2n - 1, \ldots \]

When the general term of a sequence

\[ a_1, a_2, a_3, \ldots, a_n, \ldots \]
is known, there is no need to write out the initial terms, and it is common to write only the general term enclosed in braces. Thus, (1) might be written as
\[ \{a_n\}_{n=1}^{\infty} \]
or as
\[ \{a_n\}_{n=1}^{\infty} \]
For example, here are the four sequences in Example 1 expressed in brace notation.

<table>
<thead>
<tr>
<th>SEQUENCE</th>
<th>BRACE NOTATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots</td>
<td>{\frac{n}{n+1}}_{n=1}^{\infty}</td>
</tr>
<tr>
<td>1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{2^n}, \ldots</td>
<td>{\frac{1}{2^n}}_{n=1}^{\infty}</td>
</tr>
<tr>
<td>1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, (-1)^{n+1} \frac{n}{n+1}, \ldots</td>
<td>{(-1)^{n+1} \frac{n}{n+1}}_{n=1}^{\infty}</td>
</tr>
<tr>
<td>1, 3, 5, 7, \ldots, 2n - 1, \ldots</td>
<td>{2n - 1}_{n=1}^{\infty}</td>
</tr>
</tbody>
</table>

The letter \( n \) in (1) is called the index for the sequence. It is not essential to use \( n \) for the index; any letter not reserved for another purpose can be used. For example, we might view the general term of the sequence \( a_1, a_2, a_3, \ldots \) to be the \( k \)-th term, in which case we would denote this sequence as \( \{a_k\}_{k=1}^{\infty} \). Moreover, it is not essential to start the index at 1; sometimes it is more convenient to start it at 0 (or some other integer). For example, consider the sequence
\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \ldots \]
One way to write this sequence is
\[ \{\frac{1}{2^n}\}_{n=1}^{\infty} \]
However, the general term will be simpler if we think of the initial term in the sequence as the zeroth term, in which case we can write the sequence as
\[ \{\frac{1}{2^n}\}_{n=0}^{\infty} \]

We began this section by describing a sequence as an unending succession of numbers. Although this conveys the general idea, it is not a satisfactory mathematical definition because it relies on the term “succession,” which is itself an undefined term. To motivate a precise definition, consider the sequence
\[ 2, 4, 6, 8, \ldots, 2n, \ldots \]
If we denote the general term by \( f(n) = 2n \), then we can write this sequence as
\[ f(1), f(2), f(3), \ldots, f(n), \ldots \]
which is a “list” of values of the function
\[ f(n) = 2n, \quad n = 1, 2, 3, \ldots \]
whose domain is the set of positive integers. This suggests the following definition.

**9.1.1 Definition** A sequence is a function whose domain is a set of integers.
9.1 Sequences

GRAPHS OF SEQUENCES

Since sequences are functions, it makes sense to talk about the graph of a sequence. For example, the graph of the sequence \( \{\frac{1}{n}\}_{n=1}^{\infty} \) is the graph of the equation

\[
y = \frac{1}{n}, \quad n = 1, 2, 3, \ldots
\]

Because the right side of this equation is defined only for positive integer values of \( n \), the graph consists of a succession of isolated points (Figure 9.1.1a). This is different from the graph of

\[
y = \frac{1}{x}, \quad x \geq 1
\]

which is a continuous curve (Figure 9.1.1b).

LIMIT OF A SEQUENCE

Since sequences are functions, we can inquire about their limits. However, because a sequence \( \{a_n\} \) is only defined for integer values of \( n \), the only limit that makes sense is the limit of \( a_n \) as \( n \to +\infty \). In Figure 9.1.2 we have shown the graphs of four sequences, each of which behaves differently as \( n \to +\infty \):

- The terms in the sequence \( \{n+1\}_{n=1}^{\infty} \) increase without bound.
- The terms in the sequence \( \{(-1)^{n+1}\} \) oscillate between \(-1\) and \(1\).
- The terms in the sequence \( \{n/(n+1)\} \) increase toward a “limiting value” of 1.
- The terms in the sequence \( \{1 + (\frac{-1}{2})^n\} \) also tend toward a “limiting value” of 1, but do so in an oscillatory fashion.

Informally speaking, the limit of a sequence \( \{a_n\} \) is intended to describe how \( a_n \) behaves as \( n \to +\infty \). To be more specific, we will say that a sequence \( \{a_n\} \) approaches a limit \( L \) if the terms in the sequence eventually become arbitrarily close to \( L \). Geometrically, this
means that for any positive number $\epsilon$ there is a point in the sequence after which all terms lie between the lines $y = L - \epsilon$ and $y = L + \epsilon$ (Figure 9.1.3).

The following definition makes these ideas precise.

**9.1.2 Definition** A sequence $\{a_n\}$ is said to **converge** to the limit $L$ if given any $\epsilon > 0$, there is a positive integer $N$ such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write $\lim_{n \to +\infty} a_n = L$.

A sequence that does not converge to some finite limit is said to **diverge**.

**Example 2** The first two sequences in Figure 9.1.2 diverge, and the second two converge to 1; that is,

$$\lim_{n \to +\infty} \frac{n}{n+1} = 1 \quad \text{and} \quad \lim_{n \to +\infty} \left[1 + \left(-\frac{1}{2}\right)^n\right] = 1$$

The following theorem, which we state without proof, shows that the familiar properties of limits apply to sequences. This theorem ensures that the algebraic techniques used to find limits of the form $\lim_{x \to +\infty}$ can also be used for limits of the form $\lim_{n \to +\infty}$.

**9.1.3 Theorem** Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits $L_1$ and $L_2$, respectively, and $c$ is a constant. Then:

(a) $\lim_{n \to +\infty} c = c$

(b) $\lim_{n \to +\infty} ca_n = c \lim_{n \to +\infty} a_n = cL_1$

(c) $\lim_{n \to +\infty} (a_n + b_n) = \lim_{n \to +\infty} a_n + \lim_{n \to +\infty} b_n = L_1 + L_2$

(d) $\lim_{n \to +\infty} (a_n - b_n) = \lim_{n \to +\infty} a_n - \lim_{n \to +\infty} b_n = L_1 - L_2$

(e) $\lim_{n \to +\infty} (a_nb_n) = \lim_{n \to +\infty} a_n \cdot \lim_{n \to +\infty} b_n = L_1L_2$

(f) $\lim_{n \to +\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n \to +\infty} a_n}{\lim_{n \to +\infty} b_n} = \frac{L_1}{L_2} \quad (\text{if } L_2 \neq 0)$

Additional limit properties follow from those in Theorem 9.1.3. For example, use part (e) to show that if $a_n \to L$ and $m$ is a positive integer, then $\lim_{n \to +\infty} (a_n)^m = L^m$.

If the general term of a sequence is $f(n)$, where $f(x)$ is a function defined on the entire interval $[1, +\infty)$, then the values of $f(n)$ can be viewed as “sample values” of $f(x)$ taken
9.1 Sequences

at the positive integers. Thus,

\[ \lim_{n \to \infty} \frac{n}{2n+1} = \lim_{n \to \infty} \frac{1}{2 + 1/n} = \lim_{n \to \infty} \frac{1}{2 + \frac{1}{n}} = \lim_{n \to \infty} \frac{1}{2 + \frac{1}{1/n}} = \frac{1}{2 + 0} = \frac{1}{2} \]

Thus, the sequence converges to \( \frac{1}{2} \).

Example 3. In each part, determine whether the sequence converges or diverges by examining the limit as \( n \to +\infty \).

(a) \( \left\{ \frac{n}{2n+1} \right\}_{n=1}^{+\infty} \)

(b) \( \left\{ (-1)^{n+1} \frac{n}{2n+1} \right\}_{n=1}^{+\infty} \)

(c) \( \left\{ (-1)^{n+1} \frac{1}{n} \right\}_{n=1}^{+\infty} \)

(d) \( \{8 - 2n\}_{n=1}^{+\infty} \)

Solution (a). Dividing numerator and denominator by \( n \) and using Theorem 9.1.3 yields

\[ \lim_{n \to +\infty} \frac{\frac{n}{2n+1}}{\frac{1}{2 + \frac{1}{n}}} = \lim_{n \to +\infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2 + 0} = \frac{1}{2} \]

Thus, the sequence converges to \( \frac{1}{2} \).

Solution (b). This sequence is the same as that in part (a), except for the factor of \( (-1)^{n+1} \), which oscillates between \( +1 \) and \(-1 \). Thus, the terms in this sequence oscillate between positive and negative values, with the odd-numbered terms being identical to those in part (a) and the even-numbered terms being the negatives of those in part (a). Since the sequence in part (a) has a limit of \( \frac{1}{2} \), it follows that the odd-numbered terms in this sequence approach \( \frac{1}{2} \), and the even-numbered terms approach \(-\frac{1}{2} \). Therefore, this sequence has no limit—it diverges.

Solution (c). Since \( 1/n \to 0 \), the product \( (-1)^{n+1} \frac{1}{n} \) oscillates between positive and negative values, with the odd-numbered terms approaching 0 through positive values and the even-numbered terms approaching 0 through negative values. Thus,

\[ \lim_{n \to +\infty} (-1)^{n+1} \frac{1}{n} = 0 \]

so the sequence converges to 0.

Solution (d). \( \lim_{n \to +\infty} (8 - 2n) = -\infty \), so the sequence \( \{8 - 2n\}_{n=1}^{+\infty} \) diverges.

Example 4. In each part, determine whether the sequence converges, and if so, find its limit.

(a) \( \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots \)

(b) \( 1, 2, 2^2, 2^3, \ldots, 2^n, \ldots \)

Solution. Replacing \( n \) by \( x \) in the first sequence produces the power function \( (1/2)^x \), and replacing \( n \) by \( x \) in the second sequence produces the power function \( 2^x \). Now recall that if \( 0 < b < 1 \), then \( b^x \to 0 \) as \( x \to +\infty \), and if \( b > 1 \), then \( b^x \to +\infty \) as \( x \to +\infty \) (Figure 0.5.1).
Thus, \[
\lim_{n \to \infty} \frac{1}{2^n} = 0 \quad \text{and} \quad \lim_{n \to \infty} 2^n = +\infty
\]
So, the sequence \(\{1/2^n\}\) converges to 0, but the sequence \(\{2^n\}\) diverges.

**Example 5** Find the limit of the sequence \(\left\{ \frac{n}{e^n} \right\}_{n=1}^{\infty} \).

**Solution.** The expression \(\lim_{n \to \infty} \frac{n}{e^n}\) is an indeterminate form of type \(\infty/\infty\), so L'Hôpital's rule is indicated. However, we cannot apply this rule directly to \(n/e^n\) because the functions \(n\) and \(e^n\) have been defined here only at the positive integers, and hence are not differentiable functions. To circumvent this problem we extend the domains of these functions to all real numbers, here implied by replacing \(n\) by \(x\), and apply L'Hôpital's rule to the limit of the quotient \(x/e^x\). This yields
\[
\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0
\]
from which we can conclude that
\[
\lim_{n \to +\infty} \frac{n}{e^n} = 0
\]

**Example 6** Show that \(\lim_{n \to \infty} \sqrt[n]{n} = 1\).

**Solution.**
\[
\lim_{n \to +\infty} \sqrt[n]{n} = \lim_{n \to +\infty} n^{1/n} = \lim_{n \to +\infty} e^{(1/n) \ln n} = e^0 = 1
\]

Sometimes the even-numbered and odd-numbered terms of a sequence behave sufficiently differently that it is desirable to investigate their convergence separately. The following theorem, whose proof is omitted, is helpful for that purpose.

**9.1.4 THEOREM** A sequence converges to a limit \(L\) if and only if the sequences of even-numbered terms and odd-numbered terms both converge to \(L\).

**Example 7** The sequence
\[
\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots
\]
converges to 0, since the even-numbered terms and the odd-numbered terms both converge to 0, and the sequence
\[
1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{2}, \ldots
\]
diverges, since the odd-numbered terms converge to 1 and the even-numbered terms converge to 0.

**THE SQUEEZING THEOREM FOR SEQUENCES**

The following theorem, illustrated in Figure 9.1.5, is an adaptation of the Squeezing Theorem (1.6.4) to sequences. This theorem will be useful for finding limits of sequences that cannot be obtained directly. The proof is omitted.
9.1 Sequences

9.1.5 Theorem (The Squeezing Theorem for Sequences) Let \( \{a_n\} \), \( \{b_n\} \), and \( \{c_n\} \) be sequences such that
\[
a_n \leq b_n \leq c_n \quad \text{for all values of } n \text{ beyond some index } N.
\]
If the sequences \( \{a_n\} \) and \( \{c_n\} \) have a common limit \( L \) as \( n \to +\infty \), then \( \{b_n\} \) also has the limit \( L \) as \( n \to +\infty \).

Example 8 Use numerical evidence to make a conjecture about the limit of the sequence
\[
\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}
\]
and then confirm that your conjecture is correct.

Recall that if \( n \) is a positive integer, then \( n! \) (read "n factorial") is the product of the first \( n \) positive integers. In addition, it is convenient to define \( 0! = 1 \).

Table 9.1.4

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \frac{n!}{n^n} )</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>

Solution. Table 9.1.4, which was obtained with a calculating utility, suggests that the limit of the sequence may be 0. To confirm this we need to examine the limit of
\[
a_n = \frac{n!}{n^n}
\]
as \( n \to +\infty \). Although this is an indeterminate form of type \( \frac{\infty}{\infty} \), L’Hôpital’s rule is not helpful because we have no definition of \( x! \) for values of \( x \) that are not integers. However, let us write out some of the initial terms and the general term in the sequence:

\[
a_1 = 1, \quad a_2 = \frac{1 \cdot 2}{2^2} = \frac{1}{2}, \quad a_3 = \frac{1 \cdot 2 \cdot 3}{3^3} = \frac{2}{9} < \frac{1}{3}, \quad a_4 = \frac{1 \cdot 2 \cdot 3 \cdot 4}{4^4} = \frac{3}{32} < \frac{1}{4} \cdot \ldots
\]

If \( n > 1 \), the general term of the sequence can be rewritten as
\[
a_n = \frac{1 \cdot 2 \cdots n}{n \cdot n \cdots n} = \frac{1}{n} \left( \frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n} \right)
\]
from which it follows that \( a_n \leq \frac{1}{n} \) (why?). It is now evident that
\[
0 \leq a_n \leq \frac{1}{n}
\]
However, the two outside expressions have a limit of 0 as \( n \to +\infty \); thus, the Squeezing Theorem for Sequences implies that \( a_n \to 0 \) as \( n \to +\infty \), which confirms our conjecture.

The following theorem is often useful for finding the limit of a sequence with both positive and negative terms—it states that if the sequence \( \{|a_n|\} \) that is obtained by taking the absolute value of each term in the sequence \( \{a_n\} \) converges to 0, then \( \{a_n\} \) also converges to 0.

9.1.6 Theorem If \( \lim_{n \to +\infty} |a_n| = 0 \), then \( \lim_{n \to +\infty} a_n = 0 \).

Proof Depending on the sign of \( a_n \), either \( a_n = |a_n| \) or \( a_n = -|a_n| \). Thus, in all cases we have
\[
-|a_n| \leq a_n \leq |a_n|
\]
However, the limit of the two outside terms is 0, and hence the limit of \( a_n \) is 0 by the Squeezing Theorem for Sequences.
Consider the sequence

\[ 1, -\frac{1}{2}, \frac{1}{2^2}, -\frac{1}{2^3}, \ldots, (-1)^n \frac{1}{2^n}, \ldots \]

If we take the absolute value of each term, we obtain the sequence

\[ 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \]

which, as shown in Example 4, converges to 0. Thus, from Theorem 9.1.6 we have

\[ \lim_{n \to +\infty} \left( (-1)^n \frac{1}{2^n} \right) = 0 \]

### SEQUENCES DEFINED RECURSIVELY

Some sequences do not arise from a formula for the general term, but rather from a formula or set of formulas that specify how to generate each term in the sequence from terms that precede it; such sequences are said to be defined recursively, and the defining formulas are called recursion formulas. A good example is the mechanic’s rule for approximating square roots. In Exercise 25 of Section 4.7 you were asked to show that

\[ x_1 = 1, \quad x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \]  

(2)

describes the sequence produced by Newton’s Method to approximate \( \sqrt{a} \) as a zero of the function \( f(x) = x^2 - a \). Table 9.1.5 shows the first five terms in an application of the mechanic’s rule to approximate \( \sqrt{2} \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_1 = 1 ), ( x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) )</th>
<th>DECIMAL APPROXIMATION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x_1 = 1 ) (Starting value)</td>
<td>1.00000000000</td>
</tr>
<tr>
<td>1</td>
<td>( x_2 = \frac{1}{2} \left[ 1 + \frac{2}{1} \right] = \frac{3}{2} )</td>
<td>1.50000000000</td>
</tr>
<tr>
<td>2</td>
<td>( x_3 = \frac{1}{2} \left[ \frac{17}{12} + \frac{2}{17/12} \right] = \frac{577}{408} )</td>
<td>1.41466666667</td>
</tr>
<tr>
<td>3</td>
<td>( x_4 = \frac{1}{2} \left[ \frac{577}{408} + \frac{2}{577/408} \right] = \frac{665.857}{470.832} )</td>
<td>1.41421568627</td>
</tr>
<tr>
<td>4</td>
<td>( x_5 = \frac{1}{2} \left[ \frac{665.857}{470.832} + \frac{2}{665.857/470.832} \right] = \frac{886.731}{627.013} )</td>
<td>1.41421356237</td>
</tr>
</tbody>
</table>

It would take us too far afield to investigate the convergence of sequences defined recursively, but we will conclude this section with a useful technique that can sometimes be used to compute limits of such sequences.

### Example 10

Assuming that the sequence in Table 9.1.5 converges, show that the limit is \( \sqrt{2} \).
9.1 Sequences

**Solution.** Assume that \(x_n \to L\), where \(L\) is to be determined. Since \(n + 1 \to +\infty\) as \(n \to +\infty\), it is also true that \(x_{n+1} \to L\) as \(n \to +\infty\). Thus, if we take the limit of the expression

\[ x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right) \]

as \(n \to +\infty\), we obtain

\[ L = \frac{1}{2} \left( L + \frac{2}{L} \right) \]

which can be rewritten as \(L^2 = 2\). The negative solution of this equation is extraneous because \(x_n > 0\) for all \(n\), so \(L = \sqrt{2}\).

---

**QUICK CHECK EXERCISES 9.1**

See page 607 for answers.

1. Consider the sequence 4, 6, 10, 12, ..., (a) If \(\{a_n\}_{n=1}^{\infty}\) denotes this sequence, then \(a_1 = \ldots\), \(a_2 = \ldots\), and \(a_3 = \ldots\). The general term is \(a_n = \ldots\).
   (b) If \(\{b_n\}_{n=0}^{\infty}\) denotes this sequence, then \(b_0 = \ldots\), \(b_1 = \ldots\), and \(b_2 = \ldots\). The general term is \(b_n = \ldots\).

2. What does it mean to say that a sequence \(\{a_n\}\) converges?

3. Consider sequences \(\{a_n\}\) and \(\{b_n\}\), where \(a_n \to 2\) as \(n \to +\infty\) and \(b_n = (-1)^n\). Determine which of the following sequences converge and which diverge. If a sequence converges, indicate its limit.
   (a) \(\{b_n\}\)
   (b) \(\{3a_n - 1\}\)
   (c) \(\{b_n^2\}\)
   (d) \(\{a_n + b_n\}\)
   (e) \(\left\{ \frac{1}{a_n + 3} \right\}\)
   (f) \(\left\{ \frac{b_n}{1000} \right\}\)

4. Suppose that \(\{a_n\}\), \(\{b_n\}\), and \(\{c_n\}\) are sequences such that \(a_n \leq b_n \leq c_n\) for all \(n \geq 10\), and that \(\{a_n\}\) and \(\{c_n\}\) both converge to 12. Then the _______ Theorem for Sequences implies that \(\{b_n\}\) converges to _______.

---

**EXERCISE SET 9.1**

1. In each part, find a formula for the general term of the sequence, starting with \(n = 1\).
   (a) \(1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\)
   (b) \(-1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\)
   (c) \(-1, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \ldots\)
   (d) \(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ldots\)

2. In each part, find two formulas for the general term of the sequence, one starting with \(n = 1\) and the other with \(n = 0\).
   (a) \(1, -r, r^2, -r^3, \ldots\)
   (b) \(r, -r^2, r^3, -r^4, \ldots\)

3. (a) Write out the first four terms of the sequence \(\{1 + (-1)^n\}\) starting with \(n = 0\).
   (b) Write out the first four terms of the sequence \(\{\cos n\pi\}\), starting with \(n = 0\).
   (c) Use the results in parts (a) and (b) to express the general term of the sequence 4, 0, 4, 0, ... in two different ways, starting with \(n = 0\).

4. In each part, find a formula for the general term using factorials and starting with \(n = 1\).
   (a) \(1, 2, 1 \cdot 2 \cdot 3 \cdot 4, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6, \ldots\)
   (b) \(1, 2, 1 \cdot 2 \cdot 3 \cdot 4, 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7, \ldots\)

5–6 Let \(f\) be the function \(f(x) = \cos \left( \frac{\pi}{x} \right)\) and define sequences \(\{a_n\}\) and \(\{b_n\}\) by \(a_n = f(2n)\) and \(b_n = f(2n + 1)\).

5. (a) Does \(\lim_{x \to +\infty} f(x)\) exist? Explain.
   (b) Evaluate \(a_1, a_2, a_3, a_4, \text{ and } a_5\).
   (c) Does \(\{a_n\}\) converge? If so, find its limit.

6. (a) Evaluate \(b_1, b_2, b_3, b_4, \text{ and } b_5\).
   (b) Does \(\{b_n\}\) converge? If so, find its limit.
   (c) Does \(\{f(n)\}\) converge? If so, find its limit.

7–22 Write out the first five terms of the sequence, determine whether the sequence converges, and if so find its limit.

7. \(\left\{ \frac{n}{n+2} \right\}_{n=1}^{\infty}\)
8. \(\left\{ \frac{n^2}{2n+1} \right\}_{n=1}^{\infty}\)
9. \(\left\{ (2n)_{n=1}^{\infty} \right\}\)
10. \(\left\{ \ln \left( \frac{1}{n} \right) \right\}_{n=1}^{\infty}\)
11. \(\left\{ \ln \left( \frac{n}{n+1} \right) \right\}_{n=1}^{\infty}\)
12. \(\left\{ \frac{n \sin \frac{\pi}{n}}{n} \right\}_{n=1}^{\infty}\)
13. \(\left\{ 1 + (-1)^n \right\}_{n=1}^{\infty}\)
14. \(\left\{ \frac{(-1)^n+1}{n^2} \right\}_{n=1}^{\infty}\)
15. \(\left\{ \frac{2n^3}{n^3+1} \right\}_{n=1}^{\infty}\)
16. \(\left\{ \frac{n}{2n} \right\}_{n=1}^{\infty}\)
17. \(\left\{ \frac{n+1)(n+2)}{2n^2} \right\}_{n=1}^{\infty}\)
18. \(\left\{ \frac{n^3}{4^n} \right\}_{n=1}^{\infty}\)
19. \(\left\{ n^2e^{-n} \right\}_{n=1}^{\infty}\)
20. \(\left\{ \sqrt{n^2 + 3n - n} \right\}_{n=1}^{\infty}\)
21. \( \left\{ \left( \frac{n+3}{n+1} \right)^n \right\}_{n=1}^{\infty} \)

22. \( \left\{ \left( \frac{2-n}{n} \right)^n \right\}_{n=1}^{\infty} \)

23–30 Find the general term of the sequence, starting with \( n = 1 \), determine whether the sequence converges, and if so find its limit.

23. \( \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots \)
24. \( 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \)
25. \( \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots \)
26. \(-1, 2, -3, 4, -5, \ldots \)
27. \( \left( 1 - \frac{1}{2} \right), \left( 1 - \frac{1}{3} \right), \left( 1 - \frac{1}{4} \right), \left( 1 - \frac{1}{5} \right), \ldots \)
28. \( 3, \frac{3}{2}, \frac{3}{5}, \frac{3}{10}, \ldots \)
29. \( (\sqrt{2} - \sqrt{3}), (\sqrt{3} - \sqrt{4}), (\sqrt{4} - \sqrt{5}), \ldots \)
30. \( \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots \)

31–34 True–False Determine whether the statement is true or false. Explain your answer.

31. Sequences are functions.
32. If \( \{a_n\} \) and \( \{b_n\} \) are sequences such that \( \{a_n + b_n\} \) converges, then \( \{a_n\} \) and \( \{b_n\} \) converge.
33. If \( \{a_n\} \) diverges, then \( a_n \to +\infty \) or \( a_n \to -\infty \).
34. If the graph of \( y = f(x) \) has a horizontal asymptote as \( x \to +\infty \), then the sequence \( \{f(n)\} \) converges.

35–36 Use numerical evidence to make a conjecture about the limit of the sequence, and then use the Squeezing Theorem for its limit.

35. \( \lim_{n \to +\infty} \frac{\sin^2 n}{n} \)
36. \( \lim_{n \to +\infty} \left( 1 + \frac{n}{2n} \right)^n \)

(c) Starting with \( n = 1 \), and considering the even and odd terms separately, find a formula for the general term of the sequence

\[ 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{7}, \frac{1}{9}, \frac{1}{9}, \ldots \]

(d) Determine whether the sequences in parts (a), (b), and (c) converge. For those that do, find the limit.

40. For what positive values of \( b \) does the sequence \( b, 0, b^2, 0, b^4, \ldots \) converge? Justify your answer.

41. Assuming that the sequence given in Formula (2) of this section converges, use the method of Example 10 to show that the limit of this sequence is \( \sqrt{a} \).

42. Consider the sequence

\[ a_1 = \sqrt{6} \]
\[ a_2 = \sqrt{6 + \sqrt{6}} \]
\[ a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}} \]
\[ a_4 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}} \]

(a) Find a recursion formula for \( a_{n+1} \).
(b) Assuming that the sequence converges, use the method of Example 10 to find the limit.

43. (a) A bored student enters the number 0.5 in a calculator display and then repeatedly computes the square of the number that appears in the display. Taking \( a_0 = 0.5 \), find a formula for the general term of the sequence \( \{a_n\} \) of numbers that appear in the display.
(b) Try this with a calculator and make a conjecture about the limit of \( a_n \).
(c) Confirm your conjecture by finding the limit of \( a_n \).
(d) For what values of \( a_0 \) will this procedure produce a convergent sequence?

44. Let \( f(x) = \begin{cases} 2x, & 0 \leq x < 0.5 \\ 2x - 1, & 0.5 \leq x < 1 \end{cases} \)

Does the sequence \( f(0.2), f(f(0.2)), f(f(f(0.2))), \ldots \) converge? Justify your reasoning.

45. (a) Use a graphing utility to generate the graph of the equation \( y = (2^x + 3^x)^{1/4} \), and then use the graph to make a conjecture about the limit of the sequence \( \left\{ (2^n + 3^n)^{1/4} \right\}_{n=1}^{\infty} \).
(b) Confirm your conjecture by calculating the limit.

46. Consider the sequence \( \{a_n\}_{n=1}^{\infty} \) whose general term is

\[ a_n = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (k/n)} \]

Show that \( \lim_{n \to +\infty} a_n = \ln 2 \) by interpreting \( a_n \) as the Riemann sum of a definite integral.
47. The sequence whose terms are 1, 1, 2, 3, 5, 8, 13, 21, . . . is called the Fibonacci sequence in honor of the Italian mathematician Leonardo (“Fibonacci”) da Pisa (c. 1170–1250). This sequence has the property that after starting with two 1’s, each term is the sum of the preceding two. (a) Denoting the sequence by \( \{a_n\} \) and starting with \( a_1 = 1 \) and \( a_2 = 1 \), show that

\[
\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} \text{ if } n \geq 1
\]

(b) Give a reasonable informal argument to show that if the sequence \( \{a_{n+1}/a_n\} \) converges to some limit \( L \), then the sequence \( \{a_{n+2}/a_{n+1}\} \) must also converge to \( L \).

(c) Assuming that the sequence \( \{a_{n+1}/a_n\} \) converges, show that its limit is \((1 + \sqrt{5})/2\).

48. If we accept the fact that the sequence \( \{1/n\}_{n=1}^{\infty} \) converges to the limit \( L = 0 \), then according to Definition 9.1.2, for every \( \varepsilon > 0 \) there exists a positive integer \( N \) such that \( |a_n - L| = |(1/n) - 0| < \varepsilon \) when \( n \geq N \). In each part, find the smallest possible value of \( N \) for the given value of \( \varepsilon \).

(a) \( \varepsilon = 0.5 \)  
(b) \( \varepsilon = 0.1 \)  
(c) \( \varepsilon = 0.001 \)

49. If we accept the fact that the sequence

\[
\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}
\]

converges to the limit \( L = 1 \), then according to Definition 9.1.2, for every \( \varepsilon > 0 \) there exists an integer \( N \) such that

\[
|a_n - L| = \left| \frac{n}{n+1} - 1 \right| < \varepsilon
\]

when \( n \geq N \). In each part, find the smallest value of \( N \) for the given value of \( \varepsilon \).

(a) \( \varepsilon = 0.25 \)  
(b) \( \varepsilon = 0.1 \)  
(c) \( \varepsilon = 0.001 \)

50. Use Definition 9.1.2 to prove that

(a) the sequence \( \{1/n\}_{n=1}^{\infty} \) converges to 0

(b) the sequence \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \) converges to 1.

51. Writing Discuss, with examples, various ways that a sequence could diverge.

52. Writing Discuss the convergence of the sequence \( \{r^n\} \) considering the cases \( |r| < 1, |r| > 1, r = 1, \) and \( r = -1 \) separately.

### Quick Check Answers 9.1

1. (a) 4; 10; 16; 2n + 2  
   (b) 4; 12; 20; 2n + 4  
   (d) diverges  
   (e) converges to \( \frac{1}{2} \)  
   (f) diverges

2. \( \lim_{n \to \infty} a_n \) exists

3. (a) diverges  
   (b) converges to 5  
   (c) converges to 1

4. Squeezing; 12

### 9.2 Monotone Sequences

There are many situations in which it is important to know whether a sequence converges, but the value of the limit is not relevant to the problem at hand. In this section we will study several techniques that can be used to determine whether a sequence converges.

#### Terminology

We begin with some terminology.

**9.2.1 Definition** A sequence \( \{a_n\}_{n=1}^{\infty} \) is called

- **strictly increasing** if \( a_1 < a_2 < a_3 < \cdots < a_n < \cdots \)
- **increasing** if \( a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots \)
- **strictly decreasing** if \( a_1 > a_2 > a_3 > \cdots > a_n > \cdots \)
- **decreasing** if \( a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \)

A sequence that is either increasing or decreasing is said to be **monotone**, and a sequence that is either strictly increasing or strictly decreasing is said to be **strictly monotone**.

Some examples are given in Table 9.2.1 and their corresponding graphs are shown in Figure 9.2.1. The first and second sequences in Table 9.2.1 are strictly monotone; the third...
and fourth sequences are monotone but not strictly monotone; and the fifth sequence is neither strictly monotone nor monotone.

### Table 9.2.1

<table>
<thead>
<tr>
<th>SEQUENCE</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{n}{n+1}, \ldots$</td>
<td>Strictly increasing</td>
</tr>
<tr>
<td>$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$</td>
<td>Strictly decreasing</td>
</tr>
<tr>
<td>$1, 1, 2, 2, 3, 3, \ldots$</td>
<td>Increasing; not strictly increasing</td>
</tr>
<tr>
<td>$1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots$</td>
<td>Decreasing; not strictly decreasing</td>
</tr>
<tr>
<td>$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, (-1)^{n+1} \frac{1}{n}, \ldots$</td>
<td>Neither increasing nor decreasing</td>
</tr>
</tbody>
</table>

**Figure 9.2.1**

Can a sequence be both increasing and decreasing? Explain.

### TESTING FOR MONOTONICITY

Frequently, one can **guess** whether a sequence is monotone or strictly monotone by writing out some of the initial terms. However, to be certain that the guess is correct, one must give a precise mathematical argument. Table 9.2.2 provides two ways of doing this, one based

### Table 9.2.2

<table>
<thead>
<tr>
<th>DIFFERENCE BETWEEN SUCCESSIVE TERMS</th>
<th>RATIO OF SUCCESSIVE TERMS</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{n+1} - a_n &gt; 0$</td>
<td>$a_{n+1}/a_n &gt; 1$</td>
<td>Strictly increasing</td>
</tr>
<tr>
<td>$a_{n+1} - a_n &lt; 0$</td>
<td>$a_{n+1}/a_n &lt; 1$</td>
<td>Strictly decreasing</td>
</tr>
<tr>
<td>$a_{n+1} - a_n \geq 0$</td>
<td>$a_{n+1}/a_n \geq 1$</td>
<td>Increasing</td>
</tr>
<tr>
<td>$a_{n+1} - a_n \leq 0$</td>
<td>$a_{n+1}/a_n \leq 1$</td>
<td>Decreasing</td>
</tr>
</tbody>
</table>
9.2 Monotone Sequences

on differences of successive terms and the other on ratios of successive terms. It is assumed in the latter case that the terms are positive. One must show that the specified conditions hold for all pairs of successive terms.

Example 1  Use differences of successive terms to show that
\[ \frac{n}{n+1} \]
(Figure 9.2.2) is a strictly increasing sequence.

Solution. The pattern of the initial terms suggests that the sequence is strictly increasing. To prove that this is so, let
\[ a_n = \frac{n}{n+1} \]
We can obtain \( a_{n+1} \) by replacing \( n \) by \( n + 1 \) in this formula. This yields
\[ a_{n+1} = \frac{n + 1}{(n + 1) + 1} = \frac{n + 1}{n + 2} \]
Thus, for \( n \geq 1 \)
\[ a_{n+1} - a_n = \frac{n + 1}{n + 2} - \frac{n}{n + 1} = \frac{n^2 + 2n + 1 - n^2 - 2n}{(n + 1)(n + 2)} = \frac{1}{(n + 1)(n + 2)} > 0 \]
which proves that the sequence is strictly increasing.

Example 2  Use ratios of successive terms to show that the sequence in Example 1 is strictly increasing.

Solution. As shown in the solution of Example 1,
\[ a_n = \frac{n}{n+1} \text{ and } a_{n+1} = \frac{n + 1}{n + 2} \]
Forming the ratio of successive terms we obtain
\[ \frac{a_{n+1}}{a_n} = \frac{(n + 1)/(n + 2)}{n/(n + 1)} = \frac{n + 1}{n + 2} \frac{n + 1}{n} = \frac{n^2 + 2n + 1}{n^2 + 2n} \] (1)
from which we see that \( a_{n+1}/a_n > 1 \) for \( n \geq 1 \). This proves that the sequence is strictly increasing.

Example 3  In Examples 1 and 2 we proved that the sequence
\[ \frac{n}{n+1} \]
is strictly increasing by considering the difference and ratio of successive terms. Alternatively, we can proceed as follows. Let
\[ f(x) = \frac{x}{x + 1} \]
so that the \( n \)th term in the given sequence is \( a_n = f(n) \). The function \( f \) is increasing for \( x \geq 1 \) since
\[ f'(x) = \frac{(x + 1)(1) - x(1)}{(x + 1)^2} = \frac{1}{(x + 1)^2} > 0 \]
9.2.2 DEFINITION If discarding finitely many terms from the beginning of a sequence produces a sequence with a certain property, then the original sequence is said to have that property \textbf{eventually}.

For example, although we cannot say that sequence (2) is strictly increasing, we can say that it is eventually strictly increasing.

\textbf{Example 4} Show that the sequence \(\left\{\frac{10^n}{n!}\right\}_{n=1}^{\pm \infty}\) is eventually strictly decreasing.

\textbf{Solution.} We have

\[a_n = \frac{10^n}{n!} \quad \text{and} \quad a_{n+1} = \frac{10^{n+1}}{(n+1)!}\]

so

\[
\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(n+1)!}{10^n/n!} = \frac{10^{n+1}n!}{10^n(n+1)!} = 10 \frac{n!}{(n+1)n!} = \frac{10}{n+1}
\]

From (3), \(a_{n+1}/a_n < 1\) for all \(n \geq 10\), so the sequence is eventually strictly decreasing, as confirmed by the graph in Figure 9.2.3.

\textbf{AN INTUITIVE VIEW OF CONVERGENCE}
Informally stated, the convergence or divergence of a sequence does not depend on the behavior of its \textit{initial terms}, but rather on how the terms behave \textit{eventually}. For example, the sequence

\[3, -9, -13, 17, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\]

eventually behaves like the sequence

\[1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\]

and hence has a limit of 0.

\textbf{CONVERGENCE OF MONOTONE SEQUENCES}
The following two theorems, whose proofs are discussed at the end of this section, show that a monotone sequence either converges or becomes infinite—divergence by oscillation cannot occur.
9.2 Monotone Sequences

9.2.3 Theorem If a sequence \( \{a_n\} \) is eventually increasing, then there are two possibilities:

(a) There is a constant \( M \), called an upper bound for the sequence, such that \( a_n \leq M \) for all \( n \), in which case the sequence converges to a limit \( L \) satisfying \( L \leq M \).

(b) No upper bound exists, in which case \( \lim_{n \to +\infty} a_n = +\infty \).

9.2.4 Theorem If a sequence \( \{a_n\} \) is eventually decreasing, then there are two possibilities:

(a) There is a constant \( M \), called a lower bound for the sequence, such that \( a_n \geq M \) for all \( n \), in which case the sequence converges to a limit \( L \) satisfying \( L \geq M \).

(b) No lower bound exists, in which case \( \lim_{n \to +\infty} a_n = -\infty \).

Example 5 Show that the sequence \( \left\{ \frac{10^n}{n!} \right\}_{n=1}^{+\infty} \) converges and find its limit.

Solution. We showed in Example 4 that the sequence is eventually strictly decreasing. Since all terms in the sequence are positive, it is bounded below by \( M = 0 \), and hence Theorem 9.2.4 guarantees that it converges to a nonnegative limit \( L \). However, the limit is not evident directly from the formula \( \frac{10^n}{n!} \) for the \( n \)th term, so we will need some ingenuity to obtain it.

It follows from Formula (3) of Example 4 that successive terms in the given sequence are related by the recursion formula

\[
a_{n+1} = \frac{10}{n+1} a_n
\]

where \( a_n = \frac{10^n}{n!} \). We will take the limit as \( n \to +\infty \) of both sides of (4) and use the fact that \( \lim_{n \to +\infty} a_{n+1} = \lim_{n \to +\infty} a_n = L \).

We obtain

\[
L = \lim_{n \to +\infty} a_{n+1} = \lim_{n \to +\infty} \left( \frac{10}{n+1} a_n \right) = \lim_{n \to +\infty} \frac{10}{n+1} \lim_{n \to +\infty} a_n = 0 \cdot L = 0
\]

so that

\[
L = \lim_{n \to +\infty} \frac{10^n}{n!} = 0
\]

In the exercises we will show that the technique illustrated in the last example can be adapted to obtain

\[
\lim_{n \to +\infty} \frac{x^n}{n!} = 0
\]

for any real value of \( x \) (Exercise 31). This result will be useful in our later work.

THE COMPLETENESS AXIOM

In this text we have accepted the familiar properties of real numbers without proof, and indeed, we have not even attempted to define the term real number. Although this is sufficient for many purposes, it was recognized by the late nineteenth century that the study of limits
and functions in calculus requires a precise axiomatic formulation of the real numbers analogous to the axiomatic development of Euclidean geometry. Although we will not attempt to pursue this development, we will need to discuss one of the axioms about real numbers in order to prove Theorems 9.2.3 and 9.2.4. But first we will introduce some terminology.

If \( S \) is a nonempty set of real numbers, then we call \( u \) an upper bound for \( S \) if \( u \) is greater than or equal to every number in \( S \), and we call \( l \) a lower bound for \( S \) if \( l \) is smaller than or equal to every number in \( S \). For example, if \( S \) is the set of numbers in the interval \((1, 3)\), then \( u = 10, 4, 3.2, \) and \( 3 \) are upper bounds for \( S \) and \( l = -10, 0, 0.5, \) and \( 1 \) are lower bounds for \( S \). Observe also that \( u = 3 \) is the smallest of all upper bounds and \( l = 1 \) is the largest of all lower bounds. The existence of a smallest upper bound and a largest lower bound for \( S \) is not accidental; it is a consequence of the following axiom.

\textbf{9.2.5 Axiom (The Completeness Axiom)} If a nonempty set \( S \) of real numbers has an upper bound, then it has a smallest upper bound (called the least upper bound), and if a nonempty set \( S \) of real numbers has a lower bound, then it has a largest lower bound (called the greatest lower bound).

\textbf{Proof of Theorem 9.2.3}

(a) We will prove the result for increasing sequences, and leave it for the reader to adapt the argument to sequences that are eventually increasing. Assume there exists a number \( M \) such that \( a_n \leq M \) for \( n = 1, 2, \ldots \). Then \( M \) is an upper bound for the set of terms in the sequence. By the Completeness Axiom there is a least upper bound for the terms; call it \( L \). Now let \( \epsilon \) be any positive number. Since \( L \) is the least upper bound for the terms, \( L - \epsilon \) is not an upper bound for the terms, which means that there is at least one term \( a_N \) such that

\[ a_N > L - \epsilon \]

Moreover, since \( \{a_n\} \) is an increasing sequence, we must have

\[ a_n \geq a_N > L - \epsilon \]  \hspace{1cm} (6)

when \( n \geq N \). But \( a_n \) cannot exceed \( L \) since \( L \) is an upper bound for the terms. This observation together with (6) tells us that \( L \geq a_n > L - \epsilon \) for \( n \geq N \), so all terms from the \( N \)th on are within \( \epsilon \) units of \( L \). This is exactly the requirement to have

\[ \lim_{n \to +\infty} a_n = L \]

Finally, \( L \leq M \) since \( M \) is an upper bound for the terms and \( L \) is the least upper bound. This proves part (a).

(b) If there is no number \( M \) such that \( a_n \leq M \) for \( n = 1, 2, \ldots \), then no matter how large we choose \( M \), there is a term \( a_N \) such that

\[ a_N > M \]

and, since the sequence is increasing,

\[ a_n \geq a_N > M \]

when \( n \geq N \). Thus, the terms in the sequence become arbitrarily large as \( n \) increases. That is,

\[ \lim_{n \to +\infty} a_n = +\infty \]

We omit the proof of Theorem 9.2.4 since it is similar to that of 9.2.3.
9.2 Monotone Sequences

QUICK CHECK EXERCISES 9.2 (See page 614 for answers.)

1. Classify each sequence as (I) increasing, (D) decreasing, or (N) neither increasing nor decreasing.
   1. \( \{ \frac{1}{n} \}_{n=1}^{\infty} \)
   2. \( \{ 1 - \frac{1}{n} \}_{n=1}^{\infty} \)
   3. \( \{ \frac{n}{2n+1} \}_{n=1}^{\infty} \)
   4. \( \{ \frac{n}{4n-1} \}_{n=1}^{\infty} \)
   5. \( \{ n-2^n \}_{n=1}^{\infty} \)
   6. \( \{ n-n^2 \}_{n=1}^{\infty} \)

2. Classify each sequence as (M) monotonic, (S) strictly monotonic, or (N) not monotonic.

EXERCISE SET 9.2

1–6 Use the difference \( a_{n+1} - a_n \) to show that the given sequence \( \{a_n\} \) is strictly increasing or strictly decreasing. ■
   1. \( \{ \frac{1}{n} \}_{n=1}^{\infty} \)
   2. \( \{ 1 - \frac{1}{n} \}_{n=1}^{\infty} \)
   3. \( \{ \frac{n}{2n+1} \}_{n=1}^{\infty} \)
   4. \( \{ \frac{n}{4n-1} \}_{n=1}^{\infty} \)
   5. \( \{ n-2^n \}_{n=1}^{\infty} \)
   6. \( \{ n-n^2 \}_{n=1}^{\infty} \)

7–12 Use the ratio \( a_{n+1}/a_n \) to show that the given sequence \( \{a_n\} \) is strictly increasing or strictly decreasing. ■
   7. \( \{ \frac{n}{2n+1} \}_{n=1}^{\infty} \)
   8. \( \{ \frac{2^n}{1+2^n} \}_{n=1}^{\infty} \)
   9. \( \{ n e^{-n} \}_{n=1}^{\infty} \)
   10. \( \{ \frac{10^n}{(2n)!} \}_{n=1}^{\infty} \)
   11. \( \{ \frac{n!}{n!} \}_{n=1}^{\infty} \)
   12. \( \{ \frac{5^n}{2^n} \}_{n=1}^{\infty} \)

13–16 True–False Determine whether the statement is true or false. Explain your answer. ■
   13. If \( a_{n+1} - a_n > 0 \) for all \( n \geq 1 \), then the sequence \( \{a_n\} \) is strictly increasing.
   14. A sequence \( \{a_n\} \) is monotone if \( a_{n+1} - a_n = 0 \) for all \( n \geq 1 \).
   15. Any bounded sequence converges.
   16. If \( \{a_n\} \) is eventually increasing, then \( a_{100} < a_{200} \).

17–20 Use differentiation to show that the given sequence is strictly increasing or strictly decreasing. ■
   17. \( \{ \frac{n}{2n+1} \}_{n=1}^{\infty} \)
   18. \( \{ \frac{n+2}{n+1} \}_{n=1}^{\infty} \)
   19. \( \{ \tan^{-1} n \}_{n=1}^{\infty} \)
   20. \( \{ n e^{-2n} \}_{n=1}^{\infty} \)

21–24 Show that the given sequence is eventually strictly increasing or eventually strictly decreasing. ■
   21. \( \{ 2n^2 - 7n \}_{n=1}^{\infty} \)
   22. \( \{ \frac{n}{n^2 + 10} \}_{n=1}^{\infty} \)
   23. \( \{ n! \}_{n=1}^{\infty} \)
   24. \( \{ n^2 e^{-n} \}_{n=1}^{\infty} \)

3. Since \( \frac{n}{2(n+1)} \) is strictly increasing or strictly decreasing.
   4. Since \( \frac{d}{dx}(x-8)^2 > 0 \) for all \( x \). the sequence \( \{ (x-8)^2 \} \) is strictly increasing.

25. Suppose that \( \{a_n\} \) is a monotone sequence such that \( 1 \leq a_n \leq 2 \) for all \( n \). Must the sequence converge? If not, what can you say about the limit? ■
   26. Suppose that \( \{a_n\} \) is a monotone sequence such that \( a_n \leq 2 \) for all \( n \). Must the sequence converge? If so, what can you say about the limit? ■

27. Let \( \{a_n\} \) be the sequence defined recursively by \( a_1 = \sqrt{2} \) and \( a_{n+1} = \sqrt{2 + a_n} \) for all \( n \geq 1 \).
   (a) List the first three terms of the sequence.
   (b) Show that \( a_n < 2 \) for all \( n \).
   (c) Show that \( a_{n+1} - a_n = (2 - a_n)(1 + a_n) \) for all \( n \).
   (d) Use the results in parts (b) and (c) to show that \( \{a_n\} \) is a strictly increasing sequence. [Hint: If \( x = 1 \) and \( y = 2 \), then \( x^2 - y^2 > 0 \), then it follows by factoring that \( x - y > 0 \).]
   (e) Show that \( \{a_n\} \) converges and find its limit \( L \).

28. Let \( \{a_n\} \) be the sequence defined recursively by \( a_1 = 1 \) and \( a_{n+1} = \frac{1}{2} \left[ a_n + (3/a_n) \right] \) for all \( n \geq 1 \).
   (a) Show that \( a_n \geq 3/2 \) for all \( n \geq 1 \). [Hint: What is the minimum value of \( \frac{1}{2} (x + 3/x) \) for \( x > 0 \)?]
   (b) Show that \( \{a_n\} \) is eventually decreasing. [Hint: Examine \( a_{n+1} - a_n \) or \( a_{n+1}/a_n \) and use the result in part (a).]
   (c) Show that \( \{a_n\} \) converges and find its limit \( L \).

29–30 The Beverton–Holt model is used to describe changes in a population from one generation to the next under certain assumptions. If the population in generation \( n \) is given by \( x_n \), the Beverton–Holt model predicts that the population in the next generation satisfies
   \[ x_{n+1} = \frac{RKx_n}{K + (R - 1)x_n} \]
   for some positive constants \( R \) and \( K \) with \( R > 1 \). The exercises explore some properties of this population model. ■
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29. Let \( \{x_n\} \) be the sequence of population values defined recursively by
\[ x_1 = 60, \quad \text{and for } n \geq 1, \quad x_{n+1} \text{ is given by the Beverton–Holt model with } R = 10 \text{ and } K = 300. \]
(a) List the first four terms of the sequence \( \{x_n\} \).
(b) If \( 0 < x_n < 300 \), show that \( 0 < x_{n+1} < 300 \). Conclude that \( 0 < x_n < 300 \) for \( n \geq 1 \).
(c) Show that \( \{x_n\} \) is increasing.
(d) Show that \( \{x_n\} \) converges and find its limit \( L \).

30. Let \( \{x_n\} \) be a sequence of population values defined recursively by the Beverton–Holt model for which \( x_1 > K \). Assume that the constants \( R \) and \( K \) satisfy \( R > 1 \) and \( K > 0 \).
(a) If \( x_n > K \), show that \( x_{n+1} > K \). Conclude that \( x_n > K \) for all \( n \geq 1 \).
(b) Show that \( \{x_n\} \) is decreasing.
(c) Show that \( \{x_n\} \) converges and find its limit \( L \).

31. The goal of this exercise is to establish Formula (5), namely,
\[ \lim_{n \to +\infty} \frac{x^n}{n!} = 0 \]
Let \( a_n = \frac{|x|^n}{n!} \) and observe that the case where \( x = 0 \) is obvious, so we will focus on the case where \( x \neq 0 \).
(a) Show that
\[ a_{n+1} = \frac{|x|}{n+1} a_n \]
(b) Show that the sequence \( \{a_n\} \) is eventually strictly decreasing.
(c) Show that the sequence \( \{a_n\} \) converges.

32. (a) Compare appropriate areas in the accompanying figure to deduce the following inequalities for \( n \geq 2 \):
\[ \int_1^n \ln x \, dx < \ln n! < \int_1^{n+1} \ln x \, dx \]
(b) Use the result in part (a) to show that
\[ \frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}, \quad n > 1 \]
(c) Use the Squeezing Theorem for Sequences (Theorem 9.1.5) and the result in part (b) to show that
\[ \lim_{n \to +\infty} \frac{n!}{n^e} = e^{-1} \]

33. Use the left inequality in Exercise 32(b) to show that
\[ \lim_{n \to +\infty} n\sqrt[n]{n!} = +\infty \]

34. Writing Give an example of an increasing sequence that is not eventually strictly increasing. What can you conclude about the terms of any such sequence? Explain.

35. Writing Discuss the appropriate use of “eventually” for various properties of sequences. For example, which is a useful expression: “eventually bounded” or “eventually monotone”?

Quick Check Answers 9.2
1. I; D; N; I; N
2. N; M; S
3. 1; increasing
4. 8; eventually; increasing

9.3 Infinite Series

The purpose of this section is to discuss sums that contain infinitely many terms. The most familiar examples of such sums occur in the decimal representations of real numbers. For example, when we write \( \frac{1}{3} \) in the decimal form \( 0.3333 \ldots \), we mean
\[ 1 = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots \]
which suggests that the decimal representation of \( \frac{1}{3} \) can be viewed as a sum of infinitely many real numbers.

### Sums of Infinite Series
Our first objective is to define what is meant by the “sum” of infinitely many real numbers. We begin with some terminology.
9.3 Infinite Series

9.3.1 Definition An infinite series is an expression that can be written in the form
\[ \sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots \]

The numbers \( u_1, u_2, u_3, \ldots \) are called the terms of the series.

Since it is impossible to add infinitely many numbers together directly, sums of infinite series are defined and computed by an indirect limiting process. To motivate the basic idea, consider the decimal
\[ 0.3333 \ldots \]  
This can be viewed as the infinite series
\[ 0.3 + 0.03 + 0.003 + 0.0003 + \cdots \]
or, equivalently,
\[ \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots \]

Since (1) is the decimal expansion of \( \frac{1}{3} \), any reasonable definition for the sum of an infinite series should yield \( \frac{1}{3} \) for the sum of (2). To obtain such a definition, consider the following sequence of (finite) sums:
\[ s_1 = \frac{3}{10} = 0.3 \]
\[ s_2 = \frac{3}{10} + \frac{3}{10^2} = 0.33 \]
\[ s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333 \]
\[ s_4 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} = 0.3333 \]
\[ \vdots \]

The sequence of numbers \( s_1, s_2, s_3, s_4, \ldots \) (Figure 9.3.1) can be viewed as a succession of approximations to the “sum” of the infinite series, which we want to be \( \frac{1}{3} \). As we progress through the sequence, more and more terms of the infinite series are used, and the approximations get better and better, suggesting that the desired sum of \( \frac{1}{3} \) might be the limit of this sequence of approximations. To see that this is so, we must calculate the limit of the general term in the sequence of approximations, namely,
\[ s_n = \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} \]

The problem of calculating
\[ \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} \right) \]
is complicated by the fact that both the last term and the number of terms in the sum change with \( n \). It is best to rewrite such limits in a closed form in which the number of terms does not vary, if possible. (See the discussion of closed form and open form following Example 2 in Section 5.4.) To do this, we multiply both sides of (3) by \( \frac{1}{10} \) to obtain
\[ \frac{1}{10} s_n = \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^{n+1}} \]
and then subtract (4) from (3) to obtain
\[
S_n = \frac{1}{10} S_n = \frac{3}{10} - \frac{3}{10^{n+1}}
\]
\[
\frac{9}{10} S_n = \frac{3}{10} \left(1 - \frac{1}{10^n}\right)
\]
\[
S_n = \frac{1}{3} \left(1 - \frac{1}{10^n}\right)
\]

Since \(1/10^n \to 0\) as \(n \to +\infty\), it follows that
\[
\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \frac{1}{3} \left(1 - \frac{1}{10^n}\right) = \frac{1}{3}
\]
which we denote by writing
\[
\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \cdots
\]
Motivated by the preceding example, we are now ready to define the general concept of the “sum” of an infinite series
\[
u_1 + u_2 + u_3 + \cdots + u_k + \cdots
\]
We begin with some terminology: Let \(s_n\) denote the sum of the initial terms of the series, up to and including the term with index \(n\). Thus,
\[
s_1 = u_1
\]
\[
s_2 = u_1 + u_2
\]
\[
s_3 = u_1 + u_2 + u_3
\]
\[
\vdots
\]
\[
s_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^n u_k
\]
The number \(s_n\) is called the \(n\text{th partial sum}\) of the series and the sequence \(\{s_n\}_{n=1}^{\infty}\) is called the \(sequence of partial sums\).

As \(n\) increases, the partial sum \(s_n = u_1 + u_2 + \cdots + u_n\) includes more and more terms of the series. Thus, if \(s_n\) tends toward a limit as \(n \to +\infty\), it is reasonable to view this limit as the sum of all the terms in the series. This suggests the following definition.

**9.3.2 Definition** Let \(\{s_n\}\) be the sequence of partial sums of the series
\[
u_1 + u_2 + u_3 + \cdots + u_k + \cdots
\]
If the sequence \(\{s_n\}\) converges to a limit \(S\), then the series is said to **converge** to \(S\), and \(S\) is called the **sum** of the series. We denote this by writing
\[
S = \sum_{k=1}^{\infty} u_k
\]
If the sequence of partial sums diverges, then the series is said to **diverge**. A divergent series has no sum.

**Example 1** Determine whether the series
\[
1 - 1 + 1 - 1 + 1 - 1 + \cdots
\]
converges or diverges. If it converges, find the sum.
Solution. It is tempting to conclude that the sum of the series is zero by arguing that the positive and negative terms cancel one another. However, this is not correct; the problem is that algebraic operations that hold for finite sums do not carry over to infinite series in all cases. Later, we will discuss conditions under which familiar algebraic operations can be applied to infinite series, but for this example we turn directly to Definition 9.3.2. The partial sums are

\[ s_1 = 1 \]
\[ s_2 = 1 - 1 = 0 \]
\[ s_3 = 1 - 1 + 1 = 1 \]
\[ s_4 = 1 - 1 + 1 - 1 = 0 \]

and so forth. Thus, the sequence of partial sums is

1, 0, 1, 0, 1, 0, ...

(Figure 9.3.2). Since this is a divergent sequence, the given series diverges and consequently has no sum.

GEOMETRIC SERIES

In many important series, each term is obtained by multiplying the preceding term by some fixed constant. Thus, if the initial term of the series is \( a \) and each term is obtained by multiplying the preceding term by \( r \), then the series has the form

\[
\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots \quad (a \neq 0)
\]

Such series are called geometric series, and the number \( r \) is called the ratio for the series. Here are some examples:

\[
\begin{align*}
1 + 2 + 4 + 8 + \cdots + 2^k + \cdots & \quad a = 1, r = 2 \\
\frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^k} + \cdots & \quad a = \frac{3}{10}, r = \frac{1}{10} \\
\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{k+1} \frac{1}{2^k} + \cdots & \quad a = \frac{1}{2}, r = -\frac{1}{2} \\
1 + 1 + 1 + \cdots + 1 + \cdots & \quad a = 1, r = 1 \\
1 - 1 + 1 - \cdots + (-1)^{k+1} + \cdots & \quad a = 1, r = -1 \\
1 + x + x^2 + x^3 + \cdots + x^k + \cdots & \quad a = 1, r = x
\end{align*}
\]

The following theorem is the fundamental result on convergence of geometric series.

**Theorem** A geometric series

\[
\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \quad (a \neq 0)
\]

converges if \( |r| < 1 \) and diverges if \( |r| \geq 1 \). If the series converges, then the sum is

\[
\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}
\]

**Proof** Let us treat the case \( |r| = 1 \) first. If \( r = 1 \), then the series is

\[ a + a + a + a + \cdots \]
so the $n$th partial sum is $s_n = (n + 1)a$ and

$$
\lim_{n \to \infty} s_n = \lim_{n \to \infty} (n + 1)a = \pm \infty
$$

(the sign depending on whether $a$ is positive or negative). This proves divergence. If $r = -1$, the series is

$$
a - a + a - a + \cdots
$$

so the sequence of partial sums is

$$
a, 0, a, 0, a, 0, \ldots
$$

which diverges.

Now let us consider the case where $|r| \neq 1$. The $n$th partial sum of the series is

$$
s_n = a + ar + ar^2 + \cdots + ar^n
$$

(6)

Multiplying both sides of (6) by $r$ yields

$$
rs_n = ar + ar^2 + \cdots + ar^n + ar^{n+1}
$$

(7)

and subtracting (7) from (6) gives

$$
s_n - rs_n = a - ar^{n+1}
$$

or

$$
(1 - r)s_n = a - ar^{n+1}
$$

(8)

Since $r \neq 1$ in the case we are considering, this can be rewritten as

$$
s_n = \frac{a - ar^{n+1}}{1 - r} = \frac{a}{1 - r} (1 - r^{n+1})
$$

(9)

If $|r| < 1$, then $r^{n+1}$ goes to 0 as $n \to \infty$ (can you see why?), so $\{s_n\}$ converges. From (9)

$$
\lim_{n \to \infty} s_n = \frac{a}{1 - r}
$$

If $|r| > 1$, then either $r > 1$ or $r < -1$. In the case $r > 1$, $r^{n+1}$ increases without bound as $n \to \infty$, and in the case $r < -1$, $r^{n+1}$ oscillates between positive and negative values that grow in magnitude, so $\{s_n\}$ diverges in both cases.

\[\square\]

\textbf{Example 2} In each part, determine whether the series converges, and if so find its sum.

(a) \(\sum_{k=0}^{\infty} \frac{5}{4^k}\)

(b) \(\sum_{k=1}^{\infty} 3^{2k} 5^{1-k}\)

\textbf{Solution (a).} This is a geometric series with $a = 5$ and $r = \frac{1}{4}$. Since $|r| = \frac{1}{4} < 1$, the series converges and the sum is

$$
\frac{a}{1 - r} = \frac{5}{1 - \frac{1}{4}} = \frac{20}{3}
$$

(Figure 9.3.3).

\textbf{Solution (b).} This is a geometric series in concealed form, since we can rewrite it as

$$
\sum_{k=1}^{\infty} 3^{2k} 5^{1-k} = \sum_{k=1}^{\infty} \frac{9^k}{5^k} = \sum_{k=1}^{\infty} \left(\frac{9}{5}\right)^{k-1}
$$

Since $r = \frac{9}{5} > 1$, the series diverges.

\[\square\]
9.3 Infinite Series

Example 3  Find the rational number represented by the repeating decimal
\[ 0.784784784 \ldots \]

Solution. We can write
\[ 0.784784784 \ldots = 0.784 + 0.000784 + 0.000000784 + \cdots \]
so the given decimal is the sum of a geometric series with \( a = 0.784 \) and \( r = 0.001 \). Thus,
\[ 0.784784784 \ldots = \frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{0.784}{0.999} = \frac{784}{999} \]

Example 4  In each part, find all values of \( x \) for which the series converges, and find the sum of the series for those values of \( x \).

(a) \[ \sum_{k=0}^{\infty} x^k \]
(b) \[ 3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \cdots + \frac{3(-1)^k}{2^k}x^k + \cdots \]

Solution (a). The expanded form of the series is
\[ \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots \]
The series is a geometric series with \( a = 1 \) and \( r = x \), so it converges if \( |x| < 1 \) and diverges otherwise. When the series converges its sum is
\[ \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \]

Solution (b). This is a geometric series with \( a = 3 \) and \( r = -x/2 \). It converges if \( |(-x)/2| < 1 \), or equivalently, when \( |x| < 2 \). When the series converges its sum is
\[ \sum_{k=0}^{\infty} \left( -\frac{x}{2} \right)^k = \frac{3}{1-\left(-\frac{x}{2}\right)} = \frac{6}{2+x} \]

TELESCOPING SUMS

Example 5  Determine whether the series
\[ \sum_{k=1}^{\infty} \frac{1}{k(k + 1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots \]
converges or diverges. If it converges, find the sum.

Solution. The \( n \)th partial sum of the series is
\[ s_n = \sum_{k=1}^{n} \frac{1}{k(k + 1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} \]
We will begin by rewriting \( s_n \) in closed form. This can be accomplished by using the method of partial fractions to obtain (verify)
\[ \frac{1}{k(k + 1)} = \frac{1}{k} - \frac{1}{k + 1} \]
from which we obtain the sum

\[ s_n = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots + \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 + \left( -\frac{1}{2} + \frac{1}{2} \right) + \left( -\frac{1}{3} + \frac{1}{3} \right) + \ldots + \left( -\frac{1}{n} + \frac{1}{n} \right) - \frac{1}{n+1} = 1 - \frac{1}{n+1} \]

(10)

Thus,

\[ \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1 \]

\section*{HARMONIC SERIES}

One of the most important of all diverging series is the harmonic series,

\[ \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \]

which arises in connection with the overtones produced by a vibrating musical string. It is not immediately evident that this series diverges. However, the divergence will become apparent when we examine the partial sums in detail. Because the terms in the series are all positive, the partial sums

\[ s_1 = 1, \quad s_2 = 1 + \frac{1}{2}, \quad s_3 = 1 + \frac{1}{2} + \frac{1}{3}, \quad s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \ldots \]

form a strictly increasing sequence

\[ s_1 < s_2 < s_3 < \ldots < s_n < \ldots \]

(Figure 9.3.4a). Thus, by Theorem 9.2.3 we can prove divergence by demonstrating that there is no constant \( M \) that is greater than or equal to every partial sum. To this end, we will consider some selected partial sums, namely, \( s_2, s_4, s_8, s_{16}, s_{32}, \ldots \). Note that the subscripts are successive powers of 2, so that these are the partial sums of the form \( s_{2^n} \) (Figure 9.3.4b).

These partial sums satisfy the inequalities

\[ s_2 = 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{3}{2}, \quad s_4 = s_2 + \frac{1}{4} > s_2 + \left( \frac{1}{4} + \frac{1}{4} \right) = s_2 + \frac{1}{2} > \frac{3}{2} \]

\[ s_8 = s_4 + \frac{1}{8} > s_4 + \left( \frac{1}{8} + \frac{1}{8} \right) = s_4 + \frac{1}{4} > \frac{5}{4} \]

\[ s_{16} = s_8 + \frac{1}{16} + \frac{1}{16} > s_8 + \left( \frac{1}{16} + \frac{1}{16} \right) = s_8 + \frac{1}{8} > \frac{9}{8} \]

and so on.

\[ s_{2^n} > \frac{n + 1}{2} \]

If \( M \) is any constant, we can find a positive integer \( n \) such that \( (n + 1)/2 > M \). But for this \( n \)

\[ s_{2^n} = \frac{n + 1}{2} > \frac{n + 1}{2} > M \]

so that no constant \( M \) is greater than or equal to every partial sum of the harmonic series. This proves divergence.

This divergence proof, which predates the discovery of calculus, is due to a French bishop and teacher, Nicole Oresme (1323–1382). This series eventually attracted the interest of Johann and Jakob Bernoulli (p. 700) and led them to begin thinking about the general concept of convergence, which was a new idea at that time.
9.3 Infinite Series

QUICK CHECK EXERCISES 9.3  
(See page 623 for answers.)

1. In mathematics, the terms “sequence” and “series” have different meanings: a ______ is a succession, whereas a ______ is a sum.

2. Consider the series \[ \sum_{k=1}^{\infty} \frac{1}{2^n} \]

If \( \{s_n\} \) is the sequence of partial sums for this series, then \( s_1 = \), \( s_2 = \), \( s_3 = \), \( s_4 = \), and \( s_n = \).

3. What does it mean to say that a series \( \sum u_k \) converges?

4. A geometric series is a series of the form \( \sum_{k=0}^{\infty} \frac{1}{r^k} \)

This series converges to ______ if ______. This series diverges if ______.

5. The harmonic series has the form \( \sum_{k=1}^{\infty} \frac{1}{k} \)

Does the harmonic series converge or diverge?

EXERCISE SET 9.3

1–2 In each part, find exact values for the first four partial sums, find a closed form for the \( n \)th partial sum, and determine whether the series converges by calculating the limit of the \( n \)th partial sum. If the series converges, then state its sum.

1. (a) \( 2 + \frac{2^2}{3} + \frac{2^4}{4} + \cdots + \frac{2^{2n}}{2n+1} + \cdots \)
   (b) \( \frac{1}{4} + \frac{2}{4} + \frac{2^2}{4} + \cdots + \frac{2^n}{4} + \cdots \)
   (c) \( \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(k+1)(k+2)} + \cdots \)

2. (a) \( \sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k \)  (b) \( \sum_{k=1}^{\infty} 4^{k-1} \)  (c) \( \sum_{k=1}^{\infty} \left( \frac{1}{k+3} - \frac{1}{k+4} \right) \)

3–14 Determine whether the series converges, and if so find its sum.

3. \( \sum_{k=1}^{\infty} \left( -\frac{3}{4} \right)^{k-1} \)

4. \( \sum_{k=1}^{\infty} \left( \frac{3}{2} \right)^{k+2} \)

5. \( \sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{7}{6k-1} \)

6. \( \sum_{k=1}^{\infty} \left( -\frac{3}{2} \right)^{k+1} \)

7. \( \sum_{k=1}^{\infty} \frac{1}{(k + 2)(k + 3)} \)

8. \( \sum_{k=1}^{\infty} \left( \frac{1}{2^k} - \frac{1}{2^{k+1}} \right) \)

9. \( \sum_{k=1}^{\infty} \frac{1}{9k^2 + 3k - 2} \)

10. \( \sum_{k=2}^{\infty} \frac{1}{k^2 - 1} \)

11. \( \sum_{k=3}^{\infty} \frac{1}{k - 2} \)

12. \( \sum_{k=5}^{\infty} \left( \frac{e}{5} \right)^{k-1} \)

13. \( \sum_{k=1}^{\infty} \frac{7^{k+2}}{3^{k-1}} \)

14. \( \sum_{k=1}^{\infty} 5^k \gamma^{1-k} \)

15. Match a series from one of Exercises 3, 5, 7, or 9 with the graph of its sequence of partial sums.

16. Match a series from one of Exercises 4, 6, 8, or 10 with the graph of its sequence of partial sums.
21–24 Express the repeating decimal as a fraction.

21. 0.9999…
22. 0.4444…
23. 5.373737…
24. 0.451141414…

25. Recall that a terminating decimal is a decimal whose digits are all 0 from some point on (0.5 = 0.50000…, for example). Show that a decimal of the form $0.a_1a_2…a_n$9999…, where $a_n \neq 9$, can be expressed as a terminating decimal.

### FOCUS ON CONCEPTS

26. The great Swiss mathematician Leonhard Euler (biography on p. 3) sometimes reached incorrect conclusions in his pioneering work on infinite series. For example, Euler deduced that

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$$

and

$$-1 = 1 + 2 + 4 + 8 + \cdots$$

by substituting $x = -1$ and $x = 2$ in the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

What was the problem with his reasoning?

27. A ball is dropped from a height of 10 m. Each time it strikes the ground it bounces vertically to a height that is $\frac{1}{4}$ of the preceding height. Find the total distance the ball will travel if it is assumed to bounce infinitely often.

28. The accompanying figure shows an “infinite staircase” constructed from cubes. Find the total volume of the staircase, given that the largest cube has a side of length 1 and each successive cube has a side whose length is half that of the preceding cube.

29. In each part, find a closed form for the $n$th partial sum of the series, and determine whether the series converges. If so, find its sum.

(a) $\ln \frac{1}{2} + \ln \frac{2}{3} + \ln \frac{3}{4} + \cdots + \ln \frac{k}{k+1} + \cdots$

(b) $\ln \left(1 - \frac{1}{4}\right) + \ln \left(1 - \frac{1}{9}\right) + \ln \left(1 - \frac{1}{16}\right) + \cdots$

$$+ \ln \left(1 - \frac{1}{(k+1)^2}\right) + \cdots$$

30. Use geometric series to show that

(a) $\sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}$ if $-1 < x < 1$

(b) $\sum_{k=0}^{\infty} (x-3)^k = \frac{1}{4-x}$ if $2 < x < 4$

(c) $\sum_{k=0}^{\infty} (-1)^k x^{2k} = \frac{1}{1+x^2}$ if $-1 < x < 1$.

31. In each part, find all values of $x$ for which the series converges, and find the sum of the series for those values of $x$.

(a) $x - x^3 + x^5 - x^7 + x^9 - \cdots$

(b) $\frac{1}{x^2} + \frac{2}{x^3} + \frac{4}{x^4} + \frac{8}{x^5} + \frac{16}{x^6} + \cdots$

(c) $e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + e^{-5x} + \cdots$

32. Show that for all real values of $x$

$$\sin x - \sin^2 x + \frac{1}{4} \sin^3 x - \frac{1}{8} \sin^4 x + \cdots = \frac{2 \sin x}{2 + \sin x}$$

33. Let $a_1$ be any real number, and let $\{a_n\}$ be the sequence defined recursively by

$$a_{n+1} = \frac{1}{2}(a_n + 1)$$

Make a conjecture about the limit of the sequence, and confirm your conjecture by expressing $a_n$ in terms of $a_1$ and taking the limit.

34. Show: $\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2 + k}} = 1$.

35. Show: $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{3}{2}$.

36. Show: $\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 7} + \cdots = \frac{3}{4}$.

37. Show: $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots = \frac{1}{2}$.

38. In his Treatise on the Configurations of Qualities and Motions (written in the 1350s), the French Bishop of Lisieux, Nicole Oresme, used a geometric method to find the sum of the series

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$$

In part (a) of the accompanying figure, each term in the series is represented by the area of a rectangle, and in
part (b) the configuration in part (a) has been divided into rectangles with areas \( A_1, A_2, A_3, \ldots \). Find the sum \( A_1 + A_2 + A_3 + \cdots \).

39. As shown in the accompanying figure, suppose that an angle \( \theta \) is bisected using a straightedge and compass to produce ray \( R_1 \), then the angle between \( R_1 \) and the initial side is bisected to produce ray \( R_2 \). Thereafter, rays \( R_3, R_4, R_5, \ldots \) are constructed in succession by bisecting the angle between the preceding two rays. Show that the sequence of angles that these rays make with the initial side has a limit of \( \theta/3 \).

40. In each part, use a CAS to find the sum of the series if it converges, and then confirm the result by hand calculation.

(a) \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k}{3^{k-1}} \quad \text{b) } \sum_{k=1}^{\infty} \frac{3^k}{5^k-1} \quad \text{c) } \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \]

41. Writing Discuss the similarities and differences between what it means for a sequence to converge and what it means for a series to converge.

42. Writing Read about Zeno’s dichotomy paradox in an appropriate reference work and relate the paradox in a setting that is familiar to you. Discuss a connection between the paradox and geometric series.

**Quick Check Answers 9.3**

1. sequence; series  
2. \( 1/2, 3/4, 7/8, 15/16; 1 - \frac{1}{2^n} \)  
3. The sequence of partial sums converges.  
4. \( ar^k (a \neq 0); \frac{a}{1 - r}; |r| < 1; |r| \geq 1 \)  
5. \( \frac{1}{k} \); diverge

9.4 Convergence Tests

In the last section we showed how to find the sum of a series by finding a closed form for the \( n \)th partial sum and taking its limit. However, it is relatively rare that one can find a closed form for the \( n \)th partial sum of a series, so alternative methods are needed for finding the sum of a series. One possibility is to prove that the series converges, and then to approximate the sum by a partial sum with sufficiently many terms to achieve the desired degree of accuracy. In this section we will develop various tests that can be used to determine whether a given series converges or diverges.

**The Divergence Test**

In stating general results about convergence or divergence of series, it is convenient to use the notation \( \sum u_k \) as a generic notation for a series, thus avoiding the issue of whether the sum begins with \( k = 0 \) or \( k = 1 \) or some other value. Indeed, we will see shortly that the starting index value is irrelevant to the issue of convergence. The \( k \)th term in an infinite series \( \sum u_k \) is called the **general term** of the series. The following theorem establishes
a relationship between the limit of the general term and the convergence properties of a series.

9.4.1 Theorem (The Divergence Test)

(a) If \( \lim_{k \to +\infty} u_k \neq 0 \), then the series \( \sum u_k \) diverges.

(b) If \( \lim_{k \to +\infty} u_k = 0 \), then the series \( \sum u_k \) may either converge or diverge.

Proof (a) To prove this result, it suffices to show that if the series converges, then \( \lim_{k \to +\infty} u_k = 0 \) (why?). We will prove this alternative form of (a).

Let us assume that the series converges. The general term \( u_k \) can be written as

\[
 u_k = s_k - s_{k-1}
\]

where \( s_k \) is the sum of the terms through \( u_k \) and \( s_{k-1} \) is the sum of the terms through \( u_{k-1} \). If \( S \) denotes the sum of the series, then \( \lim_{k \to +\infty} s_k = S \), and since \( (k - 1) \to +\infty \) as \( k \to +\infty \), we also have \( \lim_{k \to +\infty} s_{k-1} = S \). Thus, from (1)

\[
 \lim_{k \to +\infty} u_k = \lim_{k \to +\infty} (s_k - s_{k-1}) = S - S = 0
\]

Proof (b) To prove this result, it suffices to produce both a convergent series and a divergent series for which \( \lim_{k \to +\infty} u_k = 0 \). The following series both have this property:

\[
 \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots \quad \text{and} \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots
\]

The first is a convergent geometric series and the second is the divergent harmonic series.

The alternative form of part (a) given in the preceding proof is sufficiently important that we state it separately for future reference.

9.4.2 Theorem If the series \( \sum u_k \) converges, then \( \lim_{k \to +\infty} u_k = 0 \).

Example 1 The series

\[
 \sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{k}{k+1} + \cdots
\]

diverges since

\[
 \lim_{k \to +\infty} \frac{k}{k+1} = \lim_{k \to +\infty} \frac{1}{1 + 1/k} = 1 \neq 0
\]

Algebraic Properties of Infinite Series

For brevity, the proof of the following result is omitted.
9.4.3 **THEOREM**

(a) If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum (u_k + v_k)$ and $\sum (u_k - v_k)$ are convergent series and the sums of these series are related by

$$
\sum_{k=1}^{\infty} (u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k
$$

$$
\sum_{k=1}^{\infty} (u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k
$$

(b) If $c$ is a nonzero constant, then the series $\sum u_k$ and $\sum c u_k$ both converge or both diverge. In the case of convergence, the sums are related by

$$
\sum_{k=1}^{\infty} c u_k = c \sum_{k=1}^{\infty} u_k
$$

(c) Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer $K$, the series

$$
\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots
$$

$$
\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots
$$

both converge or both diverge.

**Example 2** Find the sum of the series

$$
\sum_{k=1}^{\infty} \left( \frac{3}{4^k} - \frac{2}{5^{k-1}} \right)
$$

**Solution.** The series

$$
\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots
$$

is a convergent geometric series ($a = \frac{3}{4}, r = \frac{1}{4}$), and the series

$$
\sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots
$$

is also a convergent geometric series ($a = 2, r = \frac{1}{5}$). Thus, from Theorems 9.4.3(a) and 9.3.3 the given series converges and

$$
\sum_{k=1}^{\infty} \left( \frac{3}{4^k} - \frac{2}{5^{k-1}} \right) = \sum_{k=1}^{\infty} \frac{3}{4^k} - \sum_{k=1}^{\infty} \frac{2}{5^{k-1}}
$$

$$
= \frac{3}{1 - \frac{1}{4}} - \frac{2}{1 - \frac{1}{5}} = \frac{3}{3} - \frac{2}{\frac{4}{5}} = -\frac{3}{2}
$$
Example 3 Determine whether the following series converge or diverge.

(a) \[ \sum_{k=1}^{\infty} \frac{5}{k} = \frac{5}{1} + \frac{5}{2} + \frac{5}{3} + \cdots \]

(b) \[ \sum_{k=10}^{\infty} \frac{1}{k} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \]

Solution. The first series is a constant times the divergent harmonic series, and hence diverges by part (b) of Theorem 9.4.3. The second series results by deleting the first nine terms from the divergent harmonic series, and hence diverges by part (c) of Theorem 9.4.3.

The Integrable Test

The expressions

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \int_{1}^{+\infty} \frac{1}{x^2} \, dx \]

are related in that the integrand in the improper integral results when the index \( k \) in the general term of the series is replaced by \( x \) and the limits of summation in the series are replaced by the corresponding limits of integration. The following theorem shows that there is a relationship between the convergence of the series and the integral.

9.4.4 Theorem (The Integrable Test) Let \( \sum u_k \) be a series with positive terms. If \( f \) is a function that is decreasing and continuous on an interval \([a, +\infty)\) and such that \( u_k = f(k) \) for all \( k \geq a \), then

\[ \begin{align*}
\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_{a}^{+\infty} f(x) \, dx
\end{align*} \]

both converge or both diverge.

The proof of the integrable test is deferred to the end of this section. However, the gist of the proof is captured in Figure 9.4.1: if the integral diverges, then so does the series (Figure 9.4.1a), and if the integral converges, then so does the series (Figure 9.4.1b).

Example 4 Show that the integrable test applies, and use the integrable test to determine whether the following series converge or diverge.

(a) \[ \sum_{k=1}^{\infty} \frac{1}{k} \]

(b) \[ \sum_{k=1}^{\infty} \frac{1}{k^2} \]

Solution (a). We already know that this is the divergent harmonic series, so the integrable test will simply illustrate another way of establishing the divergence.

Note first that the series has positive terms, so the integrable test is applicable. If we replace \( k \) by \( x \) in the general term \( 1/k \), we obtain the function \( f(x) = 1/x \), which is decreasing and continuous for \( x \geq 1 \) (as required to apply the integrable test with \( a = 1 \)). Since

\[ \int_{1}^{+\infty} \frac{1}{x} \, dx = \lim_{b \to +\infty} \int_{1}^{b} \frac{1}{x} \, dx = \lim_{b \to +\infty} [\ln b - \ln 1] = +\infty \]

the integral diverges and consequently so does the series.
9.4 Convergence Tests

**Solution (b).** Note first that the series has positive terms, so the integral test is applicable. If we replace $k$ by $x$ in the general term $1/k^2$, we obtain the function $f(x) = 1/x^2$, which is decreasing and continuous for $x \geq 1$. Since

$$
\int_1^{+\infty} \frac{1}{x^2} \, dx = \lim_{b \to +\infty} \int_1^b \frac{dx}{x^2} = \lim_{b \to +\infty} \left[ -\frac{1}{x} \right]_1^b = \lim_{b \to +\infty} \left[ 1 - \frac{1}{b} \right] = 1
$$

the integral converges and consequently the series converges by the integral test with $a = 1$.

**p-SERIES**

The series in Example 4 are special cases of a class of series called **p-series** or **hyperharmonic series**. A p-series is an infinite series of the form

$$
\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots
$$

where $p > 0$. Examples of p-series are

- \( \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots \quad (p = 1) \)
- \( \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \cdots \quad (p = 2) \)
- \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \cdots \quad (p = \frac{1}{2}) \)

The following theorem tells when a p-series converges.

**9.4.5 THEOREM (Convergence of p-Series)**

$$
\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots
$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.

**Proof.** To establish this result when $p \neq 1$, we will use the integral test.

$$
\int_1^{+\infty} \frac{1}{x^p} \, dx = \lim_{b \to +\infty} \int_1^b x^{-p} \, dx = \lim_{b \to +\infty} \left[ \frac{1}{1-p} x^{1-p} \right]_1^b = \lim_{b \to +\infty} \left[ \frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right]
$$

Assume first that $p > 1$. Then $1 - p < 0$, so $b^{1-p} \to 0$ as $b \to +\infty$. Thus, the integral converges [its value is $-1/(1-p)$] and consequently the series also converges.

Now assume that $0 < p < 1$. It follows that $1 - p > 0$ and $b^{1-p} \to +\infty$ as $b \to +\infty$, so the integral and the series diverge. The case $p = 1$ is the harmonic series, which was previously shown to diverge.

**Example 5**

$$
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \cdots
$$

diverges since it is a p-series with $p = \frac{1}{2} < 1$. 

---

**WARNING**

In part (b) of Example 4, do not erroneously conclude that the sum of the series is 1 because the value of the corresponding integral is 1. You can see that this is not so since the sum of the first two terms alone exceeds 1. Later, we will see that the sum of the series is actually $\pi^2/6$. 

---

**Proof.** To establish this result when $p \neq 1$, we will use the integral test.
PROOF OF THE INTEGRAL TEST

Before we can prove the integral test, we need a basic result about convergence of series with nonnegative terms. If \( u_1 + u_2 + u_3 + \cdots + u_k + \cdots \) is such a series, then its sequence of partial sums is increasing, that is,

\[
s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq \cdots
\]

Thus, from Theorem 9.2.3 the sequence of partial sums converges to a limit \( S \) if and only if it has some upper bound \( M \), in which case \( S \leq M \). If no upper bound exists, then the sequence of partial sums diverges. Since convergence of the sequence of partial sums corresponds to convergence of the series, we have the following theorem.

9.4.6 THEOREM If \( \sum u_k \) is a series with nonnegative terms, and if there is a constant \( M \) such that \( s_n = u_1 + u_2 + \cdots + u_n \leq M \) for every \( n \), then the series converges and the sum \( S \) satisfies \( S \leq M \). If no such \( M \) exists, then the series diverges.

In words, this theorem implies that a series with nonnegative terms converges if and only if its sequence of partial sums is bounded above.

PROOF OF THEOREM 9.4.4 We need only show that the series converges when the integral converges and that the series diverges when the integral diverges. For simplicity, we will limit the proof to the case where \( a = 1 \). Assume that \( f(x) \) satisfies the hypotheses of the theorem for \( x \geq 1 \). Since

\[
f(1) = u_1, \ f(2) = u_2, \ldots, \ f(n) = u_n, \ldots
\]

the values of \( u_1, u_2, \ldots, u_n, \ldots \) can be interpreted as the areas of the rectangles shown in Figure 9.4.2.

The following inequalities result by comparing the areas under the curve \( y = f(x) \) to the areas of the rectangles in Figure 9.4.2 for \( n > 1 \):

\[
\int_1^{n+1} f(x) \, dx < u_1 + u_2 + \cdots + u_n = s_n
\]

\[
s_n - u_1 = u_2 + u_3 + \cdots + u_n < \int_1^n f(x) \, dx
\]

These inequalities can be combined as

\[
\int_1^{n+1} f(x) \, dx < s_n < u_1 + \int_1^n f(x) \, dx
\]

If the integral \( \int_1^{+\infty} f(x) \, dx \) converges to a finite value \( L \), then from the right-hand inequality in (2)

\[
s_n < u_1 + \int_1^n f(x) \, dx < u_1 + \int_1^{+\infty} f(x) \, dx = u_1 + L
\]

Thus, each partial sum is less than the finite constant \( u_1 + L \), and the series converges by Theorem 9.4.6. On the other hand, if the integral \( \int_1^{+\infty} f(x) \, dx \) diverges, then

\[
\lim_{n \to +\infty} \int_1^{n+1} f(x) \, dx = +\infty
\]

so that from the left-hand inequality in (2), \( s_n \to +\infty \) as \( n \to +\infty \). This implies that the series also diverges.
9.4 Convergence Tests

**QUICK CHECK EXERCISES 9.4** (See page 631 for answers.)

1. The divergence test says that if \( \lim_{k \to \infty} u_k \neq 0 \), then the series \( \sum u_k \) diverges.

2. Given that \( a_1 = 3 \), \( \sum_{k=1}^{\infty} a_k = 1 \), and \( \sum_{k=1}^{\infty} b_k = 5 \), it follows that \( \sum_{k=2}^{\infty} a_k = \) \( \) and \( \sum_{k=1}^{\infty} (2a_k + b_k) = \) \( \).

3. Since \( \int_1^{\infty} (4/\sqrt{x}) \, dx = +\infty \), the \( \) test applied to the series \( \sum_{k=1}^{\infty} \) shows that this series \( \).

4. A \( p \)-series is a series of the form \( \sum_{k=1}^{\infty} \).

This series converges if \( \)\( \). This series diverges if \( \).

**EXERCISE SET 9.4**

1. Use Theorem 9.4.3 to find the sum of each series.
   (a) \( \left( \frac{1}{2} + \frac{1}{4} \right) + \left( \frac{1}{2^2} + \frac{1}{4^2} \right) + \cdots + \left( \frac{1}{2^7} + \frac{1}{4^7} \right) \)
   (b) \( \sum_{k=1}^{\infty} \left( \frac{1}{k^2} - \frac{1}{k(k+1)} \right) \)

2. Use Theorem 9.4.3 to find the sum of each series.
   (a) \( \sum_{k=2}^{\infty} \left[ \frac{1}{k^2 - 1} - \frac{7}{10^k - 1} \right] \)
   (b) \( \sum_{k=1}^{\infty} \left[ 7^{-k} 3^{k+1} - \frac{2^{k+1}}{5^k} \right] \)

3–4 For each given \( p \)-series, identify \( p \) and determine whether the series converges.

3. (a) \( \sum_{k=1}^{\infty} \frac{1}{k^3} \)
   (b) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \)
   (c) \( \sum_{k=1}^{\infty} k^{-1} \)
   (d) \( \sum_{k=1}^{\infty} k^{-2/3} \)

4. (a) \( \sum_{k=1}^{\infty} k^{-4/3} \)
   (b) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \)
   (c) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} \)
   (d) \( \sum_{k=1}^{\infty} \frac{1}{k^{7/2}} \)

5–6 Apply the divergence test and state what it tells you about the series.

5. (a) \( \sum_{k=1}^{\infty} \frac{k^2 + k + 3}{2k^2 + 1} \)
   (b) \( \sum_{k=1}^{\infty} \left( 1 + \frac{1}{k} \right)^k \)
   (c) \( \sum_{k=1}^{\infty} \cos k\pi \)
   (d) \( \sum_{k=1}^{\infty} \frac{1}{k!} \)

6. (a) \( \sum_{k=1}^{\infty} \frac{k}{\sqrt{k}} \)
   (b) \( \sum_{k=1}^{\infty} \ln k \)
   (c) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \)
   (d) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} + 3} \)

7–8 Confirm that the integral test is applicable and use it to determine whether the series converges.

7. (a) \( \int_{1}^{\infty} 5k^2 + 2 \)
   (b) \( \int_{1}^{\infty} 1 + 9k^2 \)

8. (a) \( \int_{1}^{\infty} \frac{k}{1 + k^2} \)
   (b) \( \int_{1}^{\infty} \frac{1}{(4 + 2k)^{3/2}} \)

9–24 Determine whether the series converges.

9. \( \sum_{k=1}^{\infty} \frac{1}{k + 6} \)
10. \( \sum_{k=1}^{\infty} \frac{3}{5k} \)
11. \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k + 3}} \)
12. \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k - 1}} \)
13. \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \)
14. \( \int_{1}^{\infty} \ln k \)
15. \( \sum_{k=1}^{\infty} \frac{k}{\ln(k + 1)} \)
16. \( \sum_{k=1}^{\infty} k e^{-k^2} \)
17. \( \sum_{k=1}^{\infty} \left( 1 + \frac{1}{k} \right)^{-k} \)
18. \( \sum_{k=1}^{\infty} \frac{k^2}{k^2 + 3} \)
19. \( \sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k + k^2} \)
20. \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2 + 1}} \)
21. \( \sum_{k=1}^{\infty} k^2 \sin^2 \left( \frac{1}{k} \right) \)
22. \( \sum_{k=1}^{\infty} k^2 e^{-k^3} \)
23. \( \sum_{k=1}^{\infty} 7k^{-1.01} \)
24. \( \sum_{k=1}^{\infty} \sech^2 k \)

25–26 Use the integral test to investigate the relationship between the value of \( p \) and the convergence of the series.

25. \( \sum_{k=1}^{\infty} \frac{1}{k (\ln k)^p} \)
26. \( \sum_{k=1}^{\infty} \frac{1}{(\ln k) [\ln (\ln k)]^p} \)

**FOCUS ON CONCEPTS**

27. Suppose that the series \( \sum u_k \) converges and the series \( \sum v_k \) diverges. Show that the series \( \sum (u_k + v_k) \) and \( \sum (u_k - v_k) \) both diverge. [**Hint:** Assume that \( \sum (u_k + v_k) \) converges and use Theorem 9.4.3 to obtain a contradiction.]
28. Find examples to show that if the series $\sum u_k$ and $\sum v_k$ both diverge, then the series $\sum (u_k + v_k)$ and $\sum (u_k - v_k)$ may either converge or diverge.

29–30 Use the results of Exercises 27 and 28, if needed, to determine whether each series converges or diverges.

29. (a) $\sum_{k=1}^{\infty} \left( \frac{2^{k-1}}{3} + \frac{1}{k} \right)$ (b) $\sum_{k=1}^{\infty} \left[ \frac{1}{3k + 2} - \frac{1}{k^{3/2}} \right]$  

30. (a) $\sum_{k=2}^{\infty} \left[ \frac{1}{k(\ln k)^2} - \frac{1}{k^2} \right]$ (b) $\sum_{k=2}^{\infty} \left[ ke^{-k^2} + \frac{1}{k \ln k} \right]$  

31–34 True–False Determine whether the statement is true or false. Explain your answer.

31. If $\sum u_k$ converges to $L$, then $\sum (1/u_k)$ converges to $1/L$.

32. If $\sum c u_k$ diverges for some constant $c$, then $\sum u_k$ must diverge.

33. The integral test can be used to prove that a series diverges.

34. The series $\sum_{k=1}^{\infty} \frac{1}{p^k}$ is a $p$-series.

35. Use a CAS to confirm that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

and then use these results in each part to find the sum of the series.

(a) $\sum_{k=1}^{\infty} \frac{3k^3 - 1}{k^4}$ (b) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ (c) $\sum_{k=2}^{\infty} \frac{1}{k(1-k)}$

36–40 Exercise 36 will show how a partial sum can be used to obtain upper and lower bounds on the sum of a series when the hypotheses of the integral test are satisfied. This result will be needed in Exercises 37–40.

36. (a) Let $\sum_{k=1}^{\infty} u_k$ be a convergent series with positive terms, and let $f$ be a function that is decreasing and continuous on $[n, +\infty)$ and such that $u_k = f(k)$ for $k \geq n$. Use an area argument and the accompanying figure to show that

$$\int_{n+1}^{+\infty} f(x) \, dx < \sum_{k=n+1}^{\infty} u_k < \int_{n}^{+\infty} f(x) \, dx$$

(b) Show that if $S$ is the sum of the series $\sum_{k=1}^{\infty} u_k$ and $s_n$ is the $n$th partial sum, then

$$s_n + \int_{n+1}^{+\infty} f(x) \, dx < S < s_n + \int_{n}^{+\infty} f(x) \, dx$$

37. (a) It was stated in Exercise 35 that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Show that if $s_n$ is the $n$th partial sum of this series, then

$$s_n + \frac{1}{n+1} < \frac{\pi^2}{6} < s_n + \frac{1}{n}$$

(b) Calculate $s_3$ exactly, and then use the result in part (a) to show that

$$\frac{29}{18} < \frac{\pi^2}{6} < \frac{61}{36}$$

(c) Use a calculating utility to confirm that the inequalities in part (b) are correct.

(d) Find upper and lower bounds on the error that results if the sum of the series is approximated by the 10th partial sum.

38. In each part, find upper and lower bounds on the error that results if the sum of the series is approximated by the 10th partial sum.

(a) $\sum_{k=1}^{\infty} \frac{1}{(2k + 1)^2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$ (c) $\sum_{k=1}^{\infty} \frac{k}{e^k}$

39. It was stated in Exercise 35 that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

(a) Let $s_n$ be the $n$th partial sum of the series above. Show that

$$s_n + \frac{1}{3(n+1)^3} < \frac{\pi^4}{90} < s_n + \frac{1}{3n^3}$$

(b) We can use a partial sum of the series to approximate $\pi^4/90$ to three decimal-place accuracy by capturing the
sum of the series in an interval of length 0.001 (or less).

Find the smallest value of \( n \) such that the interval containing \( \pi^4/90 \) in part (a) has a length of 0.001 or less.

(c) Approximate \( \pi^4/90 \) to three decimal places using the midpoint of an interval of width at most 0.001 that contains the sum of the series. Use a calculating utility to confirm that your answer is within 0.0005 of \( \pi^4/90 \).

We showed in Section 9.3 that the harmonic series \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges. Our objective in this problem is to demonstrate that although the partial sums of this series approach \( +\infty \), they increase extremely slowly.

(a) Use inequality (2) to show that for \( n \geq 2 \)
\[ \ln(n+1) < s_n < 1 + \ln n \]
(b) Use the inequalities in part (a) to find upper and lower bounds on the sum of the first million terms in the series.

(c) Show that the sum of the first billion terms in the series is less than 22.
(d) Find a value of \( n \) so that the sum of the first million terms is greater than 100.

41. Use a graphing utility to confirm that the integral test applies to the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-k} \), and then determine whether the series converges.

42. (a) Show that the hypotheses of the integral test are satisfied by the series \( \sum_{k=1}^{\infty} \frac{1}{k^3+1} \).
(b) Use a CAS and the integral test to confirm that the series converges.
(c) Construct a table of partial sums for \( n = 10, 20, 30, \ldots, 100 \), showing at least six decimal places.
(d) Based on your table, make a conjecture about the sum of the series to three decimal-place accuracy.
(e) Use part (b) of Exercise 36 to check your conjecture.

9.5 THE COMPARISON, RATIO, AND ROOT TESTS

In this section we will develop some more basic convergence tests for series with nonnegative terms. Later, we will use some of these tests to study the convergence of Taylor series.

THE COMPARISON TEST

We will begin with a test that is useful in its own right and is also the building block for other important convergence tests. The underlying idea of this test is to use the known convergence or divergence of a series to deduce the convergence or divergence of another series.

9.5.1 THEOREM (The Comparison Test) Let \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) be series with nonnegative terms and suppose that
\[ a_1 \leq b_1, \ a_2 \leq b_2, \ a_3 \leq b_3, \ldots, a_k \leq b_k, \ldots \]

(a) If the “bigger series” \( \sum b_k \) converges, then the “smaller series” \( \sum a_k \) also converges.
(b) If the “smaller series” \( \sum a_k \) diverges, then the “bigger series” \( \sum b_k \) also diverges.

We have left the proof of this theorem for the exercises; however, it is easy to visualize why the theorem is true by interpreting the terms in the series as areas of rectangles
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(Figure 9.5.1). The comparison test states that if the total area $\sum b_k$ is finite, then the total area $\sum a_k$ must also be finite; and if the total area $\sum a_k$ is infinite, then the total area $\sum b_k$ must also be infinite.

**Using the Comparison Test**

There are two steps required for using the comparison test to determine whether a series $\sum u_k$ with positive terms converges:

**Step 1.** Guess at whether the series $\sum u_k$ converges or diverges.

**Step 2.** Find a series that proves the guess to be correct. That is, if we guess that $\sum u_k$ diverges, we must find a divergent series whose terms are “smaller” than the corresponding terms of $\sum u_k$, and if we guess that $\sum u_k$ converges, we must find a convergent series whose terms are “bigger” than the corresponding terms of $\sum u_k$.

In most cases, the series $\sum u_k$ being considered will have its general term $u_k$ expressed as a fraction. To help with the guessing process in the first step, we have formulated two principles that are based on the form of the denominator for $u_k$. These principles sometimes suggest whether a series is likely to converge or diverge. We have called these “informal principles” because they are not intended as formal theorems. In fact, we will not guarantee that they always work. However, they work often enough to be useful.

**9.5.2 Informal Principle**  Constant terms in the denominator of $u_k$ can usually be deleted without affecting the convergence or divergence of the series.

**9.5.3 Informal Principle**  If a polynomial in $k$ appears as a factor in the numerator or denominator of $u_k$, all but the leading term in the polynomial can usually be discarded without affecting the convergence or divergence of the series.

**Example 1** Use the comparison test to determine whether the following series converge or diverge.

(a) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}}$

(b) $\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$

**Solution (a).** According to Principle 9.5.2, we should be able to drop the constant in the denominator without affecting the convergence or divergence. Thus, the given series is likely to behave like

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

which is a divergent $p$-series ($p = \frac{1}{2}$). Thus, we will guess that the given series diverges and try to prove this by finding a divergent series that is “smaller” than the given series. However, series (1) does the trick since

$$\frac{1}{\sqrt{k} - \frac{1}{2}} \geq \frac{1}{\sqrt{k}}$$

for $k = 1, 2, \ldots$

Thus, we have proved that the given series diverges.
The Comparison, Ratio, and Root Tests

Solution (b). According to Principle 9.5.3, we should be able to discard all but the leading term in the polynomial without affecting the convergence or divergence. Thus, the given series is likely to behave like

\[ \sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \]  

which converges since it is a constant times a convergent \( p \)-series (\( p = 2 \)). Thus, we will guess that the given series converges and try to prove this by finding a convergent series that is “bigger” than the given series. However, series (2) does the trick since

\[ \frac{1}{2k^2 + k} < \frac{1}{2k^2} \text{ for } k = 1, 2, \ldots \]

Thus, we have proved that the given series converges.

The Limit Comparison Test

In the last example, Principles 9.5.2 and 9.5.3 provided the guess about convergence or divergence as well as the series needed to apply the comparison test. Unfortunately, it is not always so straightforward to find the series required for comparison, so we will now consider an alternative to the comparison test that is usually easier to apply. The proof is given in Appendix D.

9.5.4 Theorem (The Limit Comparison Test) Let \( \sum a_k \) and \( \sum b_k \) be series with positive terms and suppose that

\[ \rho = \lim_{k \to +\infty} \frac{a_k}{b_k} \]

If \( \rho \) is finite and \( \rho > 0 \), then the series both converge or both diverge.

The cases where \( \rho = 0 \) or \( \rho = +\infty \) are discussed in the exercises (Exercise 56).

To use the limit comparison test we must again first guess at the convergence or divergence of \( \sum a_k \) and then find a series \( \sum b_k \) that supports our guess. The following example illustrates this principle.

Example 2 Use the limit comparison test to determine whether the following series converge or diverge.

(a) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} + 1} \)  
(b) \( \sum_{k=1}^{\infty} \frac{1}{2k^2 + k} \)  
(c) \( \sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^3 - k^3 + 2} \)

Solution (a). As in Example 1, Principle 9.5.2 suggests that the series is likely to behave like the divergent \( p \)-series (1). To prove that the given series diverges, we will apply the limit comparison test with

\[ a_k = \frac{1}{\sqrt{k} + 1} \text{ and } b_k = \frac{1}{\sqrt{k}} \]

We obtain

\[ \rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = \lim_{k \to +\infty} \frac{\sqrt{k}}{\sqrt{k} + 1} = \lim_{k \to +\infty} \frac{1}{1 + \frac{1}{\sqrt{k}}} = 1 \]

Since \( \rho \) is finite and positive, it follows from Theorem 9.5.4 that the given series diverges.
Solution (b). As in Example 1, Principle 9.5.3 suggests that the series is likely to behave like the convergent series (2). To prove that the given series converges, we will apply the limit comparison test with

\[ a_k = \frac{1}{2k^2 + k} \quad \text{and} \quad b_k = \frac{1}{2k^2} \]

We obtain

\[ \rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = \lim_{k \to +\infty} \frac{2k^2}{2k^2 + k} = \lim_{k \to +\infty} \frac{2}{2 + \frac{1}{k}} = 1 \]

Since \( \rho \) is finite and positive, it follows from Theorem 9.5.4 that the given series converges, which agrees with the conclusion reached in Example 1 using the comparison test.

Solution (c). From Principle 9.5.3, the series is likely to behave like

\[ \sum_{k=1}^{\infty} \frac{3k^3}{k^7} = \sum_{k=1}^{\infty} \frac{3}{k^4} \quad (3) \]

which converges since it is a constant times a convergent \( p \)-series. Thus, the given series is likely to converge. To prove this, we will apply the limit comparison test to series (3) and the given series. We obtain

\[ \rho = \lim_{k \to +\infty} \frac{3k^3 - 2k^2 + 4}{3k^7 - k^3 + 2} = \lim_{k \to +\infty} \frac{3k^3 - 2k^2 + 4}{3k^7 - 3k^3 + 6} = 1 \]

Since \( \rho \) is finite and nonzero, it follows from Theorem 9.5.4 that the given series converges, since (3) converges.

THE RATIO TEST

The comparison test and the limit comparison test hinge on first making a guess about convergence and then finding an appropriate series for comparison, both of which can be difficult tasks in cases where Principles 9.5.2 and 9.5.3 cannot be applied. In such cases the next test can often be used, since it works exclusively with the terms of the given series—it requires neither an initial guess about convergence nor the discovery of a series for comparison. Its proof is given in Appendix D.

9.5.5 THEOREM (The Ratio Test) Let \( \sum u_k \) be a series with positive terms and suppose that

\[ \rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} \]

(a) If \( \rho < 1 \), the series converges.
(b) If \( \rho > 1 \) or \( \rho = +\infty \), the series diverges.
(c) If \( \rho = 1 \), the series may converge or diverge, so that another test must be tried.

Example 3 Each of the following series has positive terms, so the ratio test applies. In each part, use the ratio test to determine whether the following series converge or diverge.

(a) \[ \sum_{k=1}^{\infty} \frac{1}{k!} \]  
(b) \[ \sum_{k=1}^{\infty} \frac{k}{2^k} \]  
(c) \[ \sum_{k=1}^{\infty} \frac{k^4}{k!} \]  
(d) \[ \sum_{k=3}^{\infty} \frac{(2k)!}{4^k} \]  
(e) \[ \sum_{k=1}^{\infty} \frac{1}{2k - 1} \]
9.5 The Comparison, Ratio, and Root Tests

Solution (a). The series converges, since
\[ \rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{k!}{k+1} = \lim_{k \to +\infty} \frac{1}{k+1} = 0 < 1 \]

Solution (b). The series converges, since
\[ \rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{k+1}{2k+1} = \frac{1}{2} < 1 \]

Solution (c). The series diverges, since
\[ \rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \lim_{k \to +\infty} \left(1 + \frac{1}{k}\right)^k = e > 1 \]

Solution (d). The series diverges, since
\[ \rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \lim_{k \to +\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4}\right) = \frac{1}{4} \lim_{k \to +\infty} (2k+2)(2k+1) = +\infty \]

Solution (e). The ratio test is of no help since
\[ \rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{1}{2(k+1) - 1} = \lim_{k \to +\infty} \frac{2k-1}{2k+1} = 1 \]

However, the integral test proves that the series diverges since
\[ \int_{1}^{+\infty} \frac{dx}{2x-1} = \lim_{b \to +\infty} \int_{1}^{b} \frac{dx}{2x-1} = \lim_{b \to +\infty} \frac{1}{2} \ln(2x-1) \bigg|_{1}^{b} = +\infty \]

Both the comparison test and the limit comparison test would also have worked here (verify).

The Root Test
In cases where it is difficult or inconvenient to find the limit required for the ratio test, the next test is sometimes useful. Since its proof is similar to the proof of the ratio test, we will omit it.

9.5.6 Theorem (The Root Test) Let \( \sum a_k \) be a series with positive terms and suppose that
\[ \rho = \lim_{k \to +\infty} \sqrt[k]{\frac{a_k}{k}} = \lim_{k \to +\infty} (a_k)^{1/k} \]

(a) If \( \rho < 1 \), the series converges.
(b) If \( \rho > 1 \) or \( \rho = +\infty \), the series diverges.
(c) If \( \rho = 1 \), the series may converge or diverge, so that another test must be tried.

Example 4 Use the root test to determine whether the following series converge or diverge.
\[ (a) \sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k \quad (b) \sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k} \]
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**Solution (a).** The series diverges, since
\[
\rho = \lim_{k \to +\infty} (u_k)^{1/k} = \lim_{k \to +\infty} \frac{4k - 5}{2k + 1} = 2 > 1
\]

**Solution (b).** The series converges, since
\[
\rho = \lim_{k \to +\infty} (u_k)^{1/k} = \lim_{k \to +\infty} \frac{1}{\ln(k + 1)} = 0 < 1 \quad \checkmark
\]

**QUICK CHECK EXERCISES 9.5** (See page 637 for answers.)

1–4 Select between converges or diverges to fill the first blank.

1. The series
   \[
   \sum_{k=1}^{\infty} \frac{2k^2 + 1}{2k^{3/2} - 1}
   \]
   converges by comparison with the p-series \(\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}\).

2. Since
   \[
   \lim_{k \to +\infty} \frac{(k + 1)^{3/2}}{3^k} = \lim_{k \to +\infty} \frac{1}{3} = \frac{1}{3}
   \]
   the series \(\sum_{k=1}^{\infty} \frac{k^{3/2}}{3^k}\) converges by the ______ test.

EXERCISE SET 9.5

1–2 Make a guess about the convergence or divergence of the series, and confirm your guess using the comparison test.

1. (a) \(\sum_{k=1}^{\infty} \frac{1}{5k^2 - k}\)  (b) \(\sum_{k=1}^{\infty} \frac{3}{k - \frac{7}{4}}\)

2. (a) \(\sum_{k=1}^{\infty} \frac{k + 1}{k^2 - k}\)  (b) \(\sum_{k=1}^{\infty} \frac{2}{k^2 + k}\)

3. In each part, use the comparison test to show that the series converges.
   (a) \(\sum_{k=1}^{\infty} \frac{1}{3^k + 5}\)  (b) \(\sum_{k=1}^{\infty} \frac{5 \sin^2 k}{k!}\)

4. In each part, use the comparison test to show that the series diverges.
   (a) \(\sum_{k=1}^{\infty} \frac{\ln k}{k}\)  (b) \(\sum_{k=1}^{\infty} \frac{k}{k^{3/2} - \frac{7}{2}}\)

5–10 Use the limit comparison test to determine whether the series converges.

5. \(\sum_{k=1}^{\infty} \frac{4k^3 - 2k + 6}{8k^7 + k - 8}\)

6. \(\sum_{k=1}^{\infty} \frac{1}{9k + 6}\)

7. \(\sum_{k=1}^{\infty} \frac{5}{3^k + 1}\)

8. \(\sum_{k=1}^{\infty} \frac{k(k + 3)}{(k + 1)(k + 2)(k + 5)}\)

9. \(\sum_{k=1}^{\infty} \frac{1}{\sqrt{8k^2 - 3k}}\)

10. \(\sum_{k=1}^{\infty} \frac{1}{(2k + 3)^5}\)

11–16 Use the ratio test to determine whether the series converges. If the test is inconclusive, then say so.

11. \(\sum_{k=1}^{\infty} \frac{3^k}{k!}\)

12. \(\sum_{k=1}^{\infty} \frac{4^k}{k^2}\)

13. \(\sum_{k=1}^{\infty} \frac{1}{5k}\)

14. \(\sum_{k=1}^{\infty} \frac{k}{(1/2)^k}\)

15. \(\sum_{k=1}^{\infty} \frac{k!}{k^k}\)

16. \(\sum_{k=1}^{\infty} \frac{k}{k^2 + 1}\)

17–20 Use the root test to determine whether the series converges. If the test is inconclusive, then say so.

17. \(\sum_{k=1}^{\infty} \left(\frac{3k + 2}{2k - 1}\right)^k\)

18. \(\sum_{k=1}^{\infty} \left(\frac{k}{10^6}\right)^k\)

19. \(\sum_{k=1}^{\infty} \frac{k}{3^k}\)

20. \(\sum_{k=1}^{\infty} (1 - e^{-k})^k\)

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. The limit comparison test decides convergence based on a limit of the quotient of consecutive terms in a series.

22. If \(\lim_{k \to +\infty} (u_{k+1} / u_k) = 5\), then \(\sum u_k\) diverges.
23. If \( \lim_{k \to +\infty} (k^2 u_k) = 5 \), then \( \sum u_k \) converges.

24. The root test decides convergence based on a limit of \( k \)th roots of terms in the sequence of partial sums for a series.

25–49 Use any method to determine whether the series converges.

25. \( \sum_{k=0}^{\infty} \frac{k^7}{k!} \)

26. \( \sum_{k=1}^{\infty} \frac{1}{2k+1} \)

27. \( \sum_{k=1}^{\infty} \frac{k^2}{3^k} \)

28. \( \sum_{k=1}^{\infty} \frac{k^4}{3^k} \)

29. \( \sum_{k=1}^{\infty} k^5 e^{-k} \)

30. \( \sum_{k=1}^{\infty} \frac{k^2}{k^3 + 1} \)

31. \( \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3 + 1} \)

32. \( \sum_{k=1}^{\infty} \frac{4}{2 + 3^k} \)

33. \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k + 1)}} \)

34. \( \sum_{k=1}^{\infty} \frac{2 + (-1)^k}{5^k} \)

35. \( \sum_{k=1}^{\infty} \frac{2 + \sqrt{k}}{(k + 1)^3 - 1} \)

36. \( \sum_{k=1}^{\infty} \frac{4 + |\cos x|}{k^3} \)

37. \( \sum_{k=1}^{\infty} \frac{1}{1 + \sqrt{k}} \)

38. \( \sum_{k=1}^{\infty} \frac{k!}{k^2} \)

39. \( \sum_{k=1}^{\infty} \frac{\ln k}{e^k} \)

40. \( \sum_{k=1}^{\infty} \frac{k!}{ek} \)

41. \( \sum_{k=0}^{\infty} \frac{(k + 4)!}{4^k k^4} \)

42. \( \sum_{k=1}^{\infty} \left( \frac{k}{k + 1} \right)^k \)

43. \( \sum_{k=1}^{\infty} \frac{1}{4 + 2^k} \)

44. \( \sum_{k=1}^{\infty} \frac{\sqrt{k} \ln k}{k^3 + 1} \)

45. \( \sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2} \)

46. \( \sum_{k=1}^{\infty} \frac{k^3 + k}{k! + 3} \)

47. \( \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} \)

48. \( \sum_{k=1}^{\infty} \frac{[\pi (k + 1)]^k}{k^{k+1}} \)

49. \( \sum_{k=1}^{\infty} \frac{\ln k}{3^k} \)

50. For what positive values of \( \alpha \) does the series \( \sum_{k=1}^{\infty} \left( \frac{\alpha^k}{k^\alpha} \right) \) converge?

51–52 Find the general term of the series and use the ratio test to show that the series converges.

51. \( 1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots \)

52. \( 1 + \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots \)

53. Show that \( \ln x < \sqrt{x} \) if \( x > 0 \), and use this result to investigate the convergence of

\( (a) \sum_{k=1}^{\infty} \frac{\ln k}{k^2} \)

\( (b) \sum_{k=2}^{\infty} \frac{1}{(\ln k)^2} \)

54. (a) Make a conjecture about the convergence of the series \( \sum_{k=1}^{\infty} \sin(\pi/k) \) by considering the local linear approximation of \( \sin x \) at \( x = 0 \).

(b) Try to confirm your conjecture using the limit comparison test.

55. (a) We will see later that the polynomial \( 1 - x^2/2 \) is the “local quadratic” approximation for \( \cos x \) at \( x = 0 \).

Make a conjecture about the convergence of the series

\( \sum_{k=1}^{\infty} \left[ 1 - \cos \left( \frac{1}{k} \right) \right] \)

by considering this approximation.

(b) Try to confirm your conjecture using the limit comparison test.

56. Let \( \sum a_k \) and \( \sum b_k \) be series with positive terms. Prove:
(a) If \( \lim_{k \to +\infty} (a_k/b_k) = 0 \) and \( \sum b_k \) converges, then \( \sum a_k \) converges.
(b) If \( \lim_{k \to +\infty} (a_k/b_k) = +\infty \) and \( \sum b_k \) diverges, then \( \sum a_k \) diverges.

57. Use Theorem 9.4.6 to prove the comparison test (Theorem 9.5.1).

58. Writing What does the ratio test tell you about the convergence of a geometric series? Discuss similarities between geometric series and series to which the ratio test applies.

59. Writing Given an infinite series, discuss a strategy for deciding what convergence test to use.

\( \square \) QUICK CHECK ANSWERS 9.5

1. diverges; \( 1/k^{2/3} \)
2. converges; ratio
3. diverges; ratio
4. converges; root
9.6 ALTERNATING SERIES; ABSOLUTE AND CONDITIONAL CONVERGENCE

Up to now we have focused exclusively on series with nonnegative terms. In this section we will discuss series that contain both positive and negative terms.

**ALTERNATING SERIES**

Series whose terms alternate between positive and negative, called alternating series, are of special importance. Some examples are

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots
\]

\[
\sum_{k=1}^{\infty} \left(-1\right)^{k+1} \frac{1}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots
\]

In general, an alternating series has one of the following two forms:

\[
\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots \tag{1}
\]

\[
\sum_{k=1}^{\infty} (-1)^k a_k = -a_1 + a_2 - a_3 + a_4 - \cdots \tag{2}
\]

where the \(a_k\)’s are assumed to be positive in both cases.

The following theorem is the key result on convergence of alternating series.

**9.6.1 Theorem (Alternating Series Test)**  An alternating series of either form (1) or form (2) converges if the following two conditions are satisfied:

(a) \( a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq \cdots \)

(b) \( \lim_{k \to +\infty} a_k = 0 \)

**Proof**  We will consider only alternating series of form (1). The idea of the proof is to show that if conditions (a) and (b) hold, then the sequences of even-numbered and odd-numbered partial sums converge to a common limit \(S\). It will then follow from Theorem 9.1.4 that the entire sequence of partial sums converges to \(S\).

Figure 9.6.1 shows how successive partial sums satisfying conditions (a) and (b) appear when plotted on a horizontal axis. The even-numbered partial sums

\[s_2, s_4, s_6, s_8, \ldots, s_{2n}, \ldots\]

form an increasing sequence bounded above by \(a_1\), and the odd-numbered partial sums

\[s_1, s_3, s_5, \ldots, s_{2n-1}, \ldots\]

form a decreasing sequence bounded below by 0. Thus, by Theorems 9.2.3 and 9.2.4, the even-numbered partial sums converge to some limit \(S_E\) and the odd-numbered partial sums converge to some limit \(S_O\). To complete the proof we must show that \(S_E = S_O\). But the
9.6 Alternating Series; Absolute and Conditional Convergence

(2n)-th term in the series is \(-a_{2n}\), so that \(s_{2n} - s_{2n-1} = -a_{2n}\), which can be written as

\[s_{2n-1} = s_{2n} + a_{2n}\]

However, \(2n \to +\infty\) and \(2n - 1 \to +\infty\) as \(n \to +\infty\), so that

\[S_E = \lim_{n \to +\infty} s_{2n-1} = \lim_{n \to +\infty} (s_{2n} + a_{2n}) = S_E + 0 = S_E\]

which completes the proof. ■

Example 1

Use the alternating series test to show that the following series converge.

(a) \(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\)

(b) \(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)}\)

Solution (a). The two conditions in the alternating series test are satisfied since

\[a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1}\]

and

\[\lim_{k \to +\infty} a_k = \lim_{k \to +\infty} \frac{1}{k} = 0\]

Solution (b). The two conditions in the alternating series test are satisfied since

\[\frac{a_{k+1}}{a_k} = \frac{k+4}{k+1} \cdot \frac{k+1}{k+2} \cdot \frac{k+3}{k+1} = \frac{k^2 + 4k}{k^2 + 5k + 6} = \frac{k^2 + 4k}{(k^2 + 4k) + (k + 6)} < 1\]

so

\[a_k > a_{k+1}\]

and

\[\lim_{k \to +\infty} a_k = \lim_{k \to +\infty} \frac{k + 3}{k(k+1)} = \lim_{k \to +\infty} \frac{k + 3}{1 + \frac{k^2}{k}} = 0\]

 APPROXIMATING SUMS OF ALTERNATING SERIES

The following theorem is concerned with the error that results when the sum of an alternating series is approximated by a partial sum.

9.6.2 Theorem If an alternating series satisfies the hypotheses of the alternating series test, and if \(S\) is the sum of the series, then:

(a) \(S\) lies between any two successive partial sums; that is, either

\[s_n \leq S \leq s_{n+1}\] or \(s_{n+1} \leq S \leq s_n\) (3)

depending on which partial sum is larger.

(b) If \(S\) is approximated by \(s_n\), then the absolute error \(|S - s_n|\) satisfies

\[|S - s_n| \leq a_{n+1}\] (4)

Moreover, the sign of the error \(S - s_n\) is the same as that of the coefficient of \(a_{n+1}\).
proof

We will prove the theorem for series of form (1). Referring to Figure 9.6.2 and keeping in mind our observation in the proof of Theorem 9.6.1 that the odd-numbered partial sums form a decreasing sequence converging to $S$ and the even-numbered partial sums form an increasing sequence converging to $S$, we see that successive partial sums oscillate from one side of $S$ to the other in smaller and smaller steps with the odd-numbered partial sums being larger than $S$ and the even-numbered partial sums being smaller than $S$. Thus, depending on whether $n$ is even or odd, we have

\[ s_n \leq S \leq s_{n+1} \quad \text{or} \quad s_{n+1} \leq S \leq s_n \]

which proves (3). Moreover, in either case we have

\[ |S - s_n| \leq |s_{n+1} - s_n| \]  \hspace{1cm} (5)

But $s_{n+1} - s_n = \pm a_{n+1}$ (the sign depending on whether $n$ is even or odd). Thus, it follows from (5) that $|S - s_n| \leq a_{n+1}$, which proves (4). Finally, since the odd-numbered partial sums are larger than $S$ and the even-numbered partial sums are smaller than $S$, it follows that $S - s_n$ has the same sign as the coefficient of $a_{n+1}$ (verify). ■

REMARK In words, inequality (4) states that for a series satisfying the hypotheses of the alternating series test, the magnitude of the error that results from approximating $S$ by $s_n$ is at most that of the first term that is not included in the partial sum. Also, note that if $a_1 > a_2 > \cdots > a_k > \cdots$, then inequality (4) can be strengthened to $|S - s_n| < a_{n+1}$.

Example 2 Later in this chapter we will show that the sum of the alternating harmonic series is

\[ \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots \]

This is illustrated in Figure 9.6.3.

(a) Accepting this to be so, find an upper bound on the magnitude of the error that results if $\ln 2$ is approximated by the sum of the first eight terms in the series.

(b) Find a partial sum that approximates $\ln 2$ to one decimal-place accuracy (the nearest tenth).

Solution (a). It follows from the strengthened form of (4) that

\[ |\ln 2 - s_8| < a_9 = \frac{1}{9} < 0.12 \]  \hspace{1cm} (6)

As a check, let us compute $s_8$ exactly. We obtain

\[ s_8 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} = \frac{533}{840} \]

Thus, with the help of a calculator

\[ |\ln 2 - s_8| = \left| \ln 2 - \frac{533}{840} \right| \approx 0.059 \]

This shows that the error is well under the estimate provided by upper bound (6).

Solution (b). For one decimal-place accuracy, we must choose a value of $n$ for which $|\ln 2 - s_n| \leq 0.05$. However, it follows from the strengthened form of (4) that

\[ |\ln 2 - s_n| < a_{n+1} \]

so it suffices to choose $n$ so that $a_{n+1} \leq 0.05$. 

Graph of the sequences of terms and nth partial sums for the alternating harmonic series

\[ \{s_n\} \]

\[ \{(-1)^{k+1}\frac{1}{k}\} \]

\[ \text{Graph of the sequences of terms and nth partial sums for the alternating harmonic series} \]

\[ \text{Figure 9.6.2} \]

\[ \text{Figure 9.6.3} \]
9.6 Alternating Series; Absolute and Conditional Convergence

One way to find \( n \) is to use a calculating utility to obtain numerical values for \( a_1, a_2, a_3, \ldots \) until you encounter the first value that is less than or equal to 0.05. If you do this, you will find that it is \( a_{20} = 0.05 \); this tells us that partial sum \( s_{19} \) will provide the desired accuracy. Another way to find \( n \) is to solve the inequality

\[
\frac{1}{n+1} \leq 0.05
\]

algebraically. We can do this by taking reciprocals, reversing the sense of the inequality, and then simplifying to obtain \( n \geq 19 \). Thus, \( s_{19} \) will provide the required accuracy, which is consistent with the previous result.

As Example 2 illustrates, the alternating harmonic series does not provide an efficient way to approximate \( \ln 2 \), since too many terms and hence too much computation is required to achieve reasonable accuracy. Later, we will develop better ways to approximate logarithms.

### ABSOLUTE CONVERGENCE

The series

\[
1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots
\]

does not fit in any of the categories studied so far—it has mixed signs but is not alternating. We will now develop some convergence tests that can be applied to such series.

#### 9.6.3 Definition

A series

\[
\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots
\]

is said to converge absolutely if the series of absolute values

\[
\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots
\]

converges and is said to diverge absolutely if the series of absolute values diverges.

#### Example 3

Determine whether the following series converge absolutely.

(a) \( 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \cdots \)

(b) \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \)

**Solution (a).** The series of absolute values is the convergent geometric series

\[
1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots
\]

so the given series converges absolutely.

**Solution (b).** The series of absolute values is the divergent harmonic series

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots
\]

so the given series diverges absolutely.
It is important to distinguish between the notions of convergence and absolute convergence. For example, the series in part (b) of Example 3 converges, since it is the alternating harmonic series, yet we demonstrated that it does not converge absolutely. However, the following theorem shows that if a series converges absolutely, then it converges.

\[ \sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots \]

**Theorem 9.6.4** If the series

\[ \sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots \]

converges, then so does the series

\[ \sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots \]

**Proof** We will write the series \( \sum u_k \) as

\[ \sum \left( u_k + |u_k| \right) - |u_k| \]  

We are assuming that \( \sum |u_k| \) converges, so that if we can show that \( \sum (u_k + |u_k|) \) converges, then it will follow from (7) and Theorem 9.4.3(a) that \( \sum u_k \) converges. However, the value of \( u_k + |u_k| \) is either 0 or 2\(|u_k|\), depending on the sign of \( u_k \). Thus, in all cases it is true that

\[ 0 \leq u_k + |u_k| \leq 2|u_k| \]

But \( \sum 2|u_k| \) converges, since it is a constant times the convergent series \( \sum |u_k| \); hence \( \sum (u_k + |u_k|) \) converges by the comparison test. 

**Example 4** Show that the following series converge.

(a) \[ 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} - \frac{1}{2^5} + \cdots \]

(b) \[ \sum_{k=1}^{\infty} \frac{\cos k}{k^2} \]

**Solution (a).** Observe that this is not an alternating series because the signs alternate in pairs after the first term. Thus, we have no convergence test that can be applied directly. However, we showed in Example 3(a) that the series converges absolutely, so Theorem 9.6.4 implies that it converges (Figure 9.6.4a).

**Solution (b).** With the help of a calculating utility, you will be able to verify that the signs of the terms in this series vary irregularly. Thus, we will test for absolute convergence. The series of absolute values is

\[ \sum_{k=1}^{\infty} \frac{\left| \cos k \right|}{k^2} \]

However,

\[ \frac{\left| \cos k \right|}{k^2} \leq \frac{1}{k^2} \]
But $\sum 1/k^2$ is a convergent $p$-series ($p = 2$), so the series of absolute values converges by the comparison test. Thus, the given series converges absolutely and hence converges (Figure 9.6.4b).

**CONDITIONAL CONVERGENCE**
Although Theorem 9.6.4 is a useful tool for series that converge absolutely, it provides no information about the convergence or divergence of a series that diverges absolutely. For example, consider the two series

\[
\begin{align*}
1 & \cdot \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1} \frac{1}{k} + \cdots \quad (8) \\
-1 & \cdot \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{k} - \cdots \quad (9)
\end{align*}
\]

Both of these series diverge absolutely, since in each case the series of absolute values is the divergent harmonic series

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots
\]

However, series (8) converges, since it is the alternating harmonic series, and series (9) diverges, since it is a constant times the divergent harmonic series. As a matter of terminology, a series that converges but diverges absolutely is said to **converge conditionally** (or to be **conditionally convergent**). Thus, (8) is a conditionally convergent series.

**Example 5** In Example 1(b) we used the alternating series test to show that the series

\[
\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}
\]

converges. Determine whether this series converges absolutely or converges conditionally.

**Solution.** We test the series for absolute convergence by examining the series of absolute values:

\[
\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{k+3}{k(k+1)} \right| = \sum_{k=1}^{\infty} \frac{k+3}{k(k+1)}
\]

Principle 9.5.3 suggests that the series of absolute values should behave like the divergent $p$-series with $p = 1$. To prove that the series of absolute values diverges, we will apply the limit comparison test with

\[
a_k = \frac{k+3}{k(k+1)} \quad \text{and} \quad b_k = \frac{1}{k}
\]

We obtain

\[
\rho = \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k+3}{k(k+1)} = \lim_{k \to \infty} \frac{k+3}{k+1} = 1
\]

Since $\rho$ is finite and positive, it follows from the limit comparison test that the series of absolute values diverges. Thus, the original series converges and also diverges absolutely, and so converges conditionally.

**THE RATIO TEST FOR ABSOLUTE CONVERGENCE**
Although one cannot generally infer convergence or divergence of a series from absolute divergence, the following variation of the ratio test provides a way of deducing divergence from absolute divergence in certain situations. We omit the proof.
9.6.5 **THEOREM (Ratio Test for Absolute Convergence)** Let \( \sum u_k \) be a series with nonzero terms and suppose that \( \rho = \lim_{k \to +\infty} \frac{|u_{k+1}|}{|u_k|} \).

(a) If \( \rho < 1 \), then the series \( \sum u_k \) converges absolutely and therefore converges.

(b) If \( \rho > 1 \) or if \( \rho = +\infty \), then the series \( \sum u_k \) diverges.

(c) If \( \rho = 1 \), no conclusion about convergence or absolute convergence can be drawn from this test.

**Example 6** Use the ratio test for absolute convergence to determine whether the series converges.

(a) \( \sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!} \)  

(b) \( \sum_{k=1}^{\infty} (-1)^k \frac{(2k - 1)!}{3^k} \)

**Solution (a).** Taking the absolute value of the general term \( u_k \) we obtain 

\[ |u_k| = \left| (-1)^k \frac{2^k}{k!} \right| = \frac{2^k}{k!} \]

Thus,

\[ \rho = \lim_{k \to +\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k \to +\infty} \frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} = \lim_{k \to +\infty} \frac{2}{k+1} = 0 < 1 \]

which implies that the series converges absolutely and therefore converges.

**Solution (b).** Taking the absolute value of the general term \( u_k \) we obtain 

\[ |u_k| = \left| (-1)^k \frac{(2k - 1)!}{3^k} \right| = \frac{(2k - 1)!}{3^k} \]

Thus,

\[ \rho = \lim_{k \to +\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k \to +\infty} \frac{\frac{(2k+1)!}{3^{k+1}}}{\frac{(2k - 1)!}{3^k}} = \lim_{k \to +\infty} \frac{(2k+1)!}{3 \cdot (2k - 1)!} = \frac{1}{3} \lim_{k \to +\infty} (2k)(2k+1) = +\infty \]

which implies that the series diverges.

**SUMMARY OF CONVERGENCE TESTS**

We conclude this section with a summary of convergence tests that can be used for reference. The skill of selecting a good test is developed through lots of practice. In some instances a test may be inconclusive, so another test must be tried.
### Summary of Convergence Tests

<table>
<thead>
<tr>
<th>NAME</th>
<th>STATEMENT</th>
<th>COMMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Divergence Test (9.4.1)</td>
<td>If ( \lim_{k \to +\infty} u_k \neq 0 ), then ( \sum u_k ) diverges.</td>
<td>If ( \lim_{k \to +\infty} u_k = 0 ), then ( \sum u_k ) may or may not converge.</td>
</tr>
<tr>
<td>Integral Test (9.4.4)</td>
<td>Let ( \sum_{k=1}^{\infty} u_k ) be a series with positive terms. If ( f ) is a function that is decreasing and continuous on an interval ([a, +\infty)) and such that ( u_k = f(k) ) for all ( k \geq a ), then ( \sum_{k=1}^{\infty} u_k ) and ( \int_{a}^{+\infty} f(x) , dx ) both converge or both diverge.</td>
<td>This test only applies to series that have positive terms. Try this test when ( f(x) ) is easy to integrate.</td>
</tr>
<tr>
<td>Comparison Test (9.5.1)</td>
<td>Let ( \sum_{k=1}^{\infty} a_k ) and ( \sum_{k=1}^{\infty} b_k ) be series with nonnegative terms such that ( a_1 \leq b_1, a_2 \leq b_2, \ldots, a_k \leq b_k, \ldots ). If ( \sum b_k ) converges, then ( \sum a_k ) converges, and if ( \sum a_k ) diverges, then ( \sum b_k ) diverges.</td>
<td>This test only applies to series with nonnegative terms. Try this test as a last resort; other tests are often easier to apply.</td>
</tr>
<tr>
<td>Limit Comparison Test (9.5.4)</td>
<td>Let ( \sum a_k ) and ( \sum b_k ) be series with positive terms and let ( \rho = \lim_{k \to +\infty} \frac{a_k}{b_k} ). If ( 0 &lt; \rho &lt; +\infty ), then both series converge or both diverge.</td>
<td>This is easier to apply than the comparison test, but still requires some skill in choosing the series ( \sum b_k ) for comparison.</td>
</tr>
<tr>
<td>Ratio Test (9.5.5)</td>
<td>Let ( \sum u_k ) be a series with positive terms and suppose that ( \rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} ). (a) Series converges if ( \rho &lt; 1 ). (b) Series diverges if ( \rho &gt; 1 ) or ( \rho = +\infty ). (c) The test is inconclusive if ( \rho = 1 ).</td>
<td>Try this test when ( u_k ) involves factorials or ( k )th powers.</td>
</tr>
<tr>
<td>Root Test (9.5.6)</td>
<td>Let ( \sum u_k ) be a series with positive terms and suppose that ( \rho = \lim_{k \to +\infty} \sqrt[k]{u_k} ). (a) The series converges if ( \rho &lt; 1 ). (b) The series diverges if ( \rho &gt; 1 ) or ( \rho = +\infty ). (c) The test is inconclusive if ( \rho = 1 ).</td>
<td>Try this test when ( u_k ) involves ( k )th powers.</td>
</tr>
<tr>
<td>Alternating Series Test (9.6.1)</td>
<td>If ( a_k &gt; 0 ) for ( k = 1, 2, 3, \ldots ), then the series ( a_1 - a_2 + a_3 - a_4 + \cdots ) or ( -a_1 + a_2 - a_3 + a_4 - \cdots ) converge if the following conditions hold: (a) ( a_1 \geq a_2 \geq a_3 \geq \cdots ) (b) ( \lim_{k \to +\infty} a_k = 0 )</td>
<td>This test applies only to alternating series.</td>
</tr>
<tr>
<td>Ratio Test for Absolute Convergence (9.6.5)</td>
<td>Let ( \sum u_k ) be a series with nonzero terms and suppose that ( \rho = \lim_{k \to +\infty} \frac{</td>
<td>u_{k+1}</td>
</tr>
</tbody>
</table>
QUICK CHECK EXERCISES 9.6  (See page 648 for answers.)

1. What characterizes an alternating series?
2. (a) The series
   \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \]
   converges by the alternating series test since \[ \text{and } \text{.} \]
   (b) If
   \[ S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \quad \text{and} \quad S_0 = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^2} \]
   then \[ |S - S_0| < \text{.} \]
3. Classify each sequence as conditionally convergent, absolutely convergent, or divergent.
   (a) \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} : \text{.} \]
   (b) \[ \sum_{k=1}^{\infty} (-1)^k \frac{3k - 1}{9k + 15} : \text{.} \]
   (c) \[ \sum_{k=1}^{\infty} (-1)^k \frac{1}{k(k + 2)} : \text{.} \]
   (d) \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k^2}} : \text{.} \]
4. Given that
   \[ \lim_{k \to \infty} \frac{(k + 1)^k/4^{k+1}}{k^4/4^k} = \lim_{k \to \infty} \frac{1 + 1/k}{4} = \frac{1}{4} \]
   is the series \[ \sum_{k=1}^{\infty} (-1)^k k^4/4^k \] conditionally convergent, absolutely convergent, or divergent?

EXERCISE SET 9.6  CAS

1–2 Show that the series converges by confirming that it satisfies the hypotheses of the alternating series test (Theorem 9.6.1).
1. \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k + 1} \]
2. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{3^k} \]

3–6 Determine whether the alternating series converges; justify your answer.
3. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k + 1}{3k + 1} \]
4. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k + 1}{\sqrt{k} + 1} \]
5. \[ \sum_{k=1}^{\infty} (-1)^{k+1} e^{-k} \]
6. \[ \sum_{k=3}^{\infty} (-1)^{k} \frac{\ln k}{k} \]

7–12 Use the ratio test for absolute convergence (Theorem 9.6.5) to determine whether the series converges or diverges. If the test is inconclusive, say so.
7. \[ \sum_{k=1}^{\infty} \left(\frac{3}{k}\right)^k \]
8. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k}{k^2} \]
9. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k}{k^2} \]
10. \[ \sum_{k=1}^{\infty} (-1)^{k} \frac{k}{5^k} \]
11. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^3}{e^k} \]
12. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^3}{k!} \]

13–28 Classify each series as absolutely convergent, conditionally convergent, or divergent.
13. \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k} \]
14. \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2/3}} \]
15. \[ \sum_{k=1}^{\infty} \frac{(-4)^{k}}{k^2} \]
16. \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \]
17. \[ \sum_{k=1}^{\infty} \frac{\cos k \pi}{k} \]
18. \[ \sum_{k=3}^{\infty} (-1)^{k} \frac{\ln k}{k} \]
19. \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^2}{k^3 + 1} \]
20. \[ \sum_{k=1}^{\infty} (-1)^{k} \frac{2k + 1}{k(k + 3)} \]
21. \[ \sum_{k=1}^{\infty} \frac{\sin k \pi}{2} \]
22. \[ \sum_{k=1}^{\infty} \frac{\sin k}{k^3} \]
23. \[ \sum_{k=1}^{\infty} \frac{(-1)^{k}}{k \ln k} \]
24. \[ \sum_{k=1}^{\infty} \frac{(-1)^{k}}{\sqrt{k(k + 1)}} \]
25. \[ \sum_{k=2}^{\infty} \left(\frac{1}{\ln k}\right)^k \]
26. \[ \sum_{k=1}^{\infty} \frac{k \cos k \pi}{k^2 + 1} \]
27. \[ \sum_{k=2}^{\infty} \frac{(-1)^{k+1}k!}{(2k - 1)!} \]
28. \[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k-1}}{k^2 + 1} \]

29–32 True–False  Determine whether the statement is true or false. Explain your answer.
29. An alternating series is one whose terms alternate between even and odd.
30. If a series satisfies the hypothesis of the alternating series test, then the sequence of partial sums of the series oscillates between overestimates and underestimates for the sum of the series.
31. If a series converges, then either it converges absolutely or it converges conditionally.
32. If \( \sum (u_k)^2 \) converges, then \( \sum u_k \) converges absolutely.
9.6 Alternating Series; Absolute and Conditional Convergence  647

33–36 Each series satisfies the hypotheses of the alternating series test. For the stated value of \( n \), find an upper bound on the absolute error that results if the sum of the series is approximated by the \( n \)th partial sum. ■

33. \( \sum_{k=1}^{7} \frac{(-1)^{k+1}}{k^4} ; n = 7 \)
34. \( \sum_{k=1}^{5} \frac{(-1)^{k+1}}{k!} ; n = 5 \)
35. \( \sum_{k=1}^{99} \frac{(-1)^{k+1}}{\sqrt{k}} ; n = 99 \)
36. \( \sum_{k=1}^{3} \frac{(-1)^{k+1}}{(k+1) \ln(k+1)} ; n = 3 \)

37–40 Each series satisfies the hypotheses of the alternating series test. Find a value of \( n \) for which the \( n \)th partial sum is ensured to approximate the sum of the series to the stated accuracy. ■

37. \( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} ; |\text{error}| < 0.0001 \)
38. \( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} ; |\text{error}| < 0.00001 \)
39. \( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{\sqrt{k}} ; \text{two decimal places} \)
40. \( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(k+1) \ln(k+1)} ; \text{one decimal place} \)

41–42 Find an upper bound on the absolute error that results if \( s_{10} \) is used to approximate the sum of the given geometric series. Compute \( s_{10} \) rounded to four decimal places and compare this value with the exact sum of the series. ■

41. \( \frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \frac{3}{32} + \cdots \)
42. \( \frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \cdots \)

43–46 Each series satisfies the hypotheses of the alternating series test. Approximate the sum of the series to two decimal-place accuracy. ■

43. \( 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \)
44. \( 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \)
45. \( \frac{1}{1} + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3 \cdot 2^2} - \frac{1}{4 \cdot 2^3} + \cdots \)
46. \( 1^{5} + 4 \cdot 1 - 3^5 + 4 \cdot 3^3 + 5^5 + 4 \cdot 5^3 - 7^5 + 4 \cdot 7^3 + \cdots \)

FOCUS ON CONCEPTS

47. The purpose of this exercise is to show that the error bound in part (b) of Theorem 9.6.2 can be overly conservative in certain cases.

(a) Use a CAS to confirm that
\[ \pi^2 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \]

(b) Use the CAS to show that \( |(\pi/4) - s_{25}| < 10^{-2} \).
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54. It was stated in Exercise 35 of Section 9.4 that

\[ \frac{\pi}{4} = \frac{1}{90} + \frac{1}{24} + \frac{1}{34} + \frac{1}{44} + \cdots \]

Use this to show that

\[ \frac{\pi}{4} = \frac{1}{96} + \frac{1}{34} + \frac{1}{54} + \frac{1}{74} + \cdots \]

55. Writing

Consider the series

\[ 1 - \frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{2}{5} - \frac{1}{6} + \cdots \]

Determine whether this series converges and use this series as an example in a discussion of the importance of hypotheses (a) and (b) of the alternating series test (Theorem 9.6.1).

56. Writing

Discuss the ways that conditional convergence is "conditional." In particular, describe how one could rearrange the terms of a conditionally convergent series \( \sum u_k \) so that the resulting series diverges, either to \( +\infty \) or to \( -\infty \).

[Hint: See Exercise 50.]
and its first two derivatives match those of \( f \) at 0. Thus, we want
\[
p(0) = f(0), \quad p'(0) = f'(0), \quad p''(0) = f''(0)
\] (3)
But the values of \( p(0), p'(0), \) and \( p''(0) \) are as follows:
\[
p(x) = c_0 + c_1x + c_2x^2 \quad p(0) = c_0
\]
\[
p'(x) = c_1 + 2c_2x \quad p'(0) = c_1
\]
\[
p''(x) = 2c_2 \quad p''(0) = 2c_2
\]
Thus, it follows from (3) that
\[
c_0 = f(0), \quad c_1 = f'(0), \quad c_2 = \frac{f''(0)}{2}
\]
and substituting these in (2) yields the following formula for the local quadratic approximation of \( f \) at \( x = 0 \):
\[
f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2
\] (4)

Example 1  Find the local linear and quadratic approximations of \( e^x \) at \( x = 0 \), and graph \( e^x \) and the two approximations together.

Solution. If we let \( f(x) = e^x \), then \( f'(x) = f''(x) = e^x \); and hence
\[
f(0) = f'(0) = f''(0) = e^0 = 1
\]
Thus, from (4) the local quadratic approximation of \( e^x \) at \( x = 0 \) is
\[
e^x \approx 1 + x + \frac{x^2}{2}
\]
and the local linear approximation (which is the linear part of the local quadratic approximation) is
\[
e^x \approx 1 + x
\]
The graphs of \( e^x \) and the two approximations are shown in Figure 9.7.2. As expected, the local quadratic approximation is more accurate than the local linear approximation near \( x = 0 \). ▶

MACLAURIN POLYNOMIALS
It is natural to ask whether one can improve on the accuracy of a local quadratic approximation by using a polynomial of degree 3. Specifically, one might look for a polynomial of degree 3 with the property that its value and the values of its first three derivatives match
Chapter 9 / Infinite Series

ear approximation of \( f(x) \) ≈ \( x \)

quadratic approximations at Local linear approximations and local

ify that \( f(x) \) approximation at a function \( f \)

MacLaurin polynomials for \( p \approx 2 \) is the local quadratic ap-

tron \( x_0 \). Ver-

Thus, to satisfy (6) we must have

Thus, we want a polynomial

such that

But

Thus, we want a polynomial

such that

We will begin by solving this problem in the case where \( x_0 = 0 \). Thus, we want a polynomial

such that

Note that the polynomial in (7) has the property that its value and the values of its first \( n \) derivatives match the values of \( f \) and its first \( n \) derivatives at \( x = 0 \).
Newton. His major mathematical work began in 1811 with a series of brilliant solutions to some difficult outstanding problems. In 1814 he wrote a treatise on integrals that was to become the basis for modern complex variable theory; in 1816 there followed a classic paper on wave propagation in liquids that won a prize from the French Academy; and in 1822 he wrote a paper that formed the basis of modern elasticity theory. Cauchy’s mathematical contributions for the next 35 years were brilliant and staggering in quantity, over 700 papers filling 26 modern volumes. Cauchy’s work initiated the era of modern analysis. He brought to mathematics standards of precision and rigor undreamed of by Leibniz and Newton.

**Example 2**  Find the Maclaurin polynomials $p_0$, $p_1$, $p_2$, $p_3$, and $p_n$ for $e^x$.

**Solution.** Let $f(x) = e^x$. Thus,

$$f'(x) = f''(x) = f'''(x) = \cdots = f^{(n)}(x) = e^x$$

and

$$f(0) = f'(0) = f''(0) = f'''(0) = \cdots = f^{(n)}(0) = e^0 = 1$$

Therefore,

$$p_0(x) = f(0) = 1$$

$$p_1(x) = f(0) + f'(0)x = 1 + x$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2!} = 1 + x + \frac{1}{2}x^2$$

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Figure 9.7.3 shows the graph of $e^x$ (in blue) and the graph of the first four Maclaurin polynomials. Note that the graphs of $p_1(x)$, $p_2(x)$, and $p_3(x)$ are virtually indistinguishable from the graph of $e^x$ near $x = 0$, so these polynomials are good approximations of $e^x$ for $x$ near 0. However, the farther $x$ is from 0, the poorer these approximations become. This is typical of the Maclaurin polynomials for a function $f(x)$; they provide good approximations of $f(x)$ near 0, but the accuracy diminishes as $x$ progresses away from 0. It is usually the case that the higher the degree of the polynomial, the larger the interval on which it provides a specified accuracy. Accuracy issues will be investigated later.

**Augustin Louis Cauchy** (1789–1857) French mathematician. Cauchy’s early education was acquired from his father, a barrister and master of the classics. Cauchy entered L’Ecole Polytechnique in 1805 to study engineering, but because of poor health, was advised to concentrate on mathematics. His major mathematical work began in 1811 with a series of brilliant solutions to some difficult outstanding problems. In 1814 he wrote a treatise on integrals that was to become the basis for modern complex variable theory; in 1816 there followed a classic paper on wave propagation in liquids that won a prize from the French Academy; and in 1822 he wrote a paper that formed the basis of modern elasticity theory. Cauchy’s mathematical contributions for the next 35 years were brilliant and staggering in quantity, over 700 papers filling 26 modern volumes. Cauchy’s work initiated the era of modern analysis. He brought to mathematics standards of precision and rigor undreamed of by Leibniz and Newton.

Cauchy’s life was inextricably tied to the political upheavals of the time. A strong partisan of the Bourbons, he left his wife and children in 1830 to follow the Bourbon king Charles X into exile. For his loyalty he was made a baron by the ex-king. Cauchy eventually returned to France, but refused to accept a university position until the government waived its requirement that he take a loyalty oath.

It is difficult to get a clear picture of the man. Devoutly Catholic, he sponsored charitable work for unwed mothers, criminals, and relief for Ireland. Yet other aspects of his life cast him in an unfavorable light. The Norwegian mathematician Abel described him as, “mad, infinitely Catholic, and bigoted.” Some writers praise his teaching, yet others say he rambled incoherently and, according to a report of the day, he once devoted an entire lecture to extracting the square root of seventeen to ten decimal places by a method well known to his students. In any event, Cauchy is undeniably one of the greatest minds in the history of science.

Example 3  Find the $n$th Maclaurin polynomials for 

(a) $\sin x$  

(b) $\cos x$

Solution (a).  In the Maclaurin polynomials for $\sin x$, only the odd powers of $x$ appear explicitly. To see this, let $f(x) = \sin x$; thus,

\[
\begin{align*}
 f(x) &= \sin x \quad f(0) = 0 \\
 f'(x) &= \cos x \quad f'(0) = 1 \\
 f''(x) &= -\sin x \quad f''(0) = 0 \\
 f'''(x) &= -\cos x \quad f'''(0) = -1
\end{align*}
\]

Since $f^{(4)}(x) = \sin x = f(x)$, the pattern $0, 1, 0, -1$ will repeat as we evaluate successive derivatives at $0$. Therefore, the successive Maclaurin polynomials for $\sin x$ are

\[
\begin{align*}
 p_0(x) &= 0 \\
p_1(x) &= 0 + x \\
p_2(x) &= 0 + x + 0 \\
p_3(x) &= 0 + x + 0 - \frac{x^3}{3!} \\
p_4(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 \\
p_5(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} \\
p_6(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 \\
p_7(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!}
\end{align*}
\]

Because of the zero terms, each even-order Maclaurin polynomial [after $p_0(x)$] is the same as the preceding odd-order Maclaurin polynomial. That is,

\[
p_{2k+1}(x) = p_{2k+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (k = 0, 1, 2, \ldots)
\]

The graphs of $\sin x$, $p_1(x)$, $p_3(x)$, $p_5(x)$, and $p_7(x)$ are shown in Figure 9.7.4.

Solution (b).  In the Maclaurin polynomials for $\cos x$, only the even powers of $x$ appear explicitly; the computations are similar to those in part (a). The reader should be able to show that

\[
\begin{align*}
 p_0(x) &= p_1(x) = 1 \\
p_2(x) &= p_3(x) = 1 - \frac{x^2}{2!} \\
p_4(x) &= p_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\
p_6(x) &= p_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}
\end{align*}
\]

In general, the Maclaurin polynomials for $\cos x$ are given by

\[
p_{2k}(x) = p_{2k+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} \quad (k = 0, 1, 2, \ldots)
\]

The graphs of $\cos x$, $p_0(x)$, $p_2(x)$, $p_4(x)$, and $p_6(x)$ are shown in Figure 9.7.5.  \( \blacksquare \)
9.7 Maclaurin and Taylor Polynomials

**TAYLOR POLYNOMIALS**

Up to now we have focused on approximating a function $f$ in the vicinity of $x = 0$. Now we will consider the more general case of approximating $f$ in the vicinity of an arbitrary domain value $x_0$. The basic idea is the same as before; we want to find an $n$th-degree polynomial $p$ with the property that its value and the values of its first $n$ derivatives match those of $f$ at $x_0$. However, rather than expressing $p(x)$ in powers of $x$, it will simplify the computations if we express it in powers of $x - x_0$; that is,

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n$$  \hspace{1cm} (8)

We will leave it as an exercise for you to imitate the computations used in the case where $x_0 = 0$ to show that

$$c_0 = f(x_0), \hspace{1cm} c_1 = f'(x_0), \hspace{1cm} c_2 = \frac{f''(x_0)}{2!}, \hspace{1cm} c_3 = \frac{f'''(x_0)}{3!}, \ldots, \hspace{1cm} c_n = \frac{f^{(n)}(x_0)}{n!}$$

Substituting these values in (8) we obtain a polynomial called the $n$th **Taylor polynomial about $x = x_0$ for $f$**.

**Example 4** Find the first four Taylor polynomials for $\ln x$ about $x = 2$.

**Solution.** Let $f(x) = \ln x$. Thus,

$$f(x) = \ln x \hspace{1cm} f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \hspace{1cm} f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \hspace{1cm} f''(2) = -\frac{1}{4}$$

$$f'''(x) = \frac{2}{x^3} \hspace{1cm} f'''(2) = \frac{1}{4}$$

The Maclaurin polynomials are the special cases of the Taylor polynomials in which $x_0 = 0$. Thus, theorems about Taylor polynomials also apply to Maclaurin polynomials.

Brook Taylor (1685–1731) English mathematician. Taylor was born of well-to-do parents. Musicians and artists were entertained frequently in the Taylor home, which undoubtedly had a lasting influence on him. In later years, Taylor published a definitive work on the mathematical theory of perspective and obtained major mathematical results about the vibrations of strings. There also exists an unpublished work, *On Musick*, that was intended to be part of a joint paper with Isaac Newton. Taylor’s life was scarred with unhappiness, ill-health, and tragedy. Because his first wife was not rich enough to suit his father, the two men argued bitterly and parted ways. Subsequently, his wife died in childbirth. Then, after he remarried, his second wife also died in childbirth, though his daughter survived. Taylor’s most productive period was from 1714 to 1719, during which time he wrote on a wide range of subjects—magnetism, capillary action, thermometers, perspective, and calculus. In his final years, Taylor devoted his writing efforts to religion and philosophy. According to Taylor, the results that bear his name were motivated by coffeehouse conversations about works of Newton on planetary motion and works of Halley (“Halley’s comet”) on roots of polynomials. Unfortunately, Taylor’s writing style was so terse and hard to understand that he never received credit for many of his innovations.

Substituting in (9) with \( x_0 = 2 \) yields
\[
p_0(x) = f(2) = \ln 2
\]
\[
p_1(x) = f(2) + f'(2)(x - 2) = \ln 2 + \frac{1}{2}(x - 2)
\]
\[
p_2(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 = \ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2
\]
\[
p_3(x) = f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \frac{f'''(2)}{3!}(x - 2)^3
\]
\[
= \ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3
\]
The graph of \( \ln x \) (in blue) and its first four Taylor polynomials about \( x = 2 \) are shown in Figure 9.7.6. As expected, these polynomials produce their best approximations of \( \ln x \) near 2.

**SIGMA NOTATION FOR TAYLOR AND MACLAURIN POLYNOMIALS**

Frequently, we will want to express Formula (9) in sigma notation. To do this, we use the notation \( f^{(k)}(x_0) \) to denote the \( k \)th derivative of \( f \) at \( x = x_0 \), and we make the convention that \( f^{(0)}(x_0) \) denotes \( f(x_0) \). This enables us to write
\[
\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = f(x_0) + f'(x_0)(x - x_0)
\]
\[
+ \frac{f''(x_0)}{2!} (x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n
\]
In particular, we can write the \( n \)th Maclaurin polynomial for \( f(x) \) as
\[
\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n
\]

**Example 5** Find the \( n \)th Maclaurin polynomial for
\[
\frac{1}{1 - x}
\]
and express it in sigma notation.

**Solution.** Let \( f(x) = 1/(1 - x) \). The values of \( f \) and its first \( k \) derivatives at \( x = 0 \) are as follows:
\[
f(x) = \frac{1}{1 - x} \quad f(0) = 1 = 0! \]
\[
f'(x) = \frac{1}{(1 - x)^2} \quad f'(0) = 1 = 1! \]
\[
f''(x) = \frac{2}{(1 - x)^3} \quad f''(0) = 2 = 2! \]
\[
f'''(x) = \frac{3 \cdot 2}{(1 - x)^4} \quad f'''(0) = 3! \]
\[
f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2}{(1 - x)^5} \quad f^{(4)}(0) = 4! \]
\[
\vdots \quad \vdots \]
\[
f^{(k)}(x) = \frac{k!}{(1 - x)^{k+1}} \quad f^{(k)}(0) = k! \]

**TECHNOLOGY MASTERY**

Computer algebra systems have commands for generating Taylor polynomials of any specified degree. If you have a CAS, use it to find some of the Maclaurin and Taylor polynomials in Examples 3, 4, and 5.
9.7 Maclaurin and Taylor Polynomials

Thus, substituting $f^{(k)}(0) = k!$ into Formula (11) yields the $n$th Maclaurin polynomial for $1/(1-x)$:

$$p_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n \quad (n = 0, 1, 2, \ldots)$$

Example 6  Find the $n$th Taylor polynomial for $1/x$ about $x = 1$ and express it in sigma notation.

Solution. Let $f(x) = 1/x$. The computations are similar to those in Example 5. We leave it for you to show that

$$f(1) = 1, \quad f'(1) = -1, \quad f''(1) = 2!, \quad f'''(1) = -3!, \quad f^{(4)}(1) = 4!, \ldots, \quad f^{(k)}(1) = (-1)^k k!$$

Thus, substituting $f^{(k)}(1) = (-1)^k k!$ into Formula (10) with $x_0 = 1$ yields the $n$th Taylor polynomial for $1/x$:

$$\sum_{k=0}^{n} (-1)^k k (x-1)^k = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n$$

THE nTH REMAINDER

It will be convenient to have a notation for the error in the approximation $f(x) \approx p_n(x)$. Accordingly, we will let $R_n(x)$ denote the difference between $f(x)$ and its $n$th Taylor polynomial; that is,

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

This can also be written as

$$f(x) = p_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + R_n(x)$$

The function $R_n(x)$ is called the $n$th remainder for the Taylor series of $f$, and Formula (13) is called Taylor’s formula with remainder.

Finding a bound for $R_n(x)$ gives an indication of the accuracy of the approximation $p_n(x) \approx f(x)$. The following theorem, which is proved in Appendix D, provides such a bound.

9.7.4 Theorem (The Remainder Estimation Theorem)  If the function $f$ can be differentiated $n + 1$ times on an interval containing the number $x_0$, and if $M$ is an upper bound for $|f^{(n+1)}(x)|$ on the interval, that is, $|f^{(n+1)}(x)| \leq M$ for all $x$ in the interval, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-x_0|^{n+1}$$

for all $x$ in the interval.
Example 7 Use an $n$th Maclaurin polynomial for $e^x$ to approximate $e$ to five decimal-place accuracy.

Solution. We note first that the exponential function $e^x$ has derivatives of all orders for every real number $x$. From Example 2, the $n$th Maclaurin polynomial for $e^x$ is

$$
\sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}
$$

from which we have

$$
e \approx e^1 \approx \sum_{k=0}^{n} \frac{1^k}{k!} = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}
$$

Thus, our problem is to determine how many terms to include in a Maclaurin polynomial for $e^x$ to achieve five decimal-place accuracy; that is, we want to choose $n$ so that the absolute value of the $n$th remainder at $x = 1$ satisfies

$$
|R_n(1)| \leq 0.000005
$$

To determine $n$ we use the Remainder Estimation Theorem with $f(x) = e^x$, $x = 1$, $x_0 = 0$, and the interval $[0, 1]$. In this case it follows from (14) that

$$
|R_n(1)| \leq \frac{M}{(n+1)!} \cdot |1 - 0|^{n+1} = \frac{M}{(n+1)!}
$$

where $M$ is an upper bound on the value of $f^{(n+1)}(x) = e^x$ for $x$ in the interval $[0, 1]$. However, $e^x$ is an increasing function, so its maximum value on the interval $[0, 1]$ occurs at $x = 1$; that is, $e^x \leq e$ on this interval. Thus, we can take $M = e$ in (15) to obtain

$$
|R_n(1)| \leq \frac{e}{(n+1)!}
$$

Unfortunately, this inequality is not very useful because it involves $e$, which is the very quantity we are trying to approximate. However, if we accept that $e < 3$, then we can replace (16) with the following less precise, but more easily applied, inequality:

$$
|R_n(1)| \leq \frac{3}{(n+1)!}
$$

Thus, we can achieve five decimal-place accuracy by choosing $n$ so that

$$
\frac{3}{(n+1)!} \leq 0.000005 \quad \text{or} \quad (n+1)! \geq 600,000
$$

Since $9! = 362,880$ and $10! = 3,628,800$, the smallest value of $n$ that meets this criterion is $n = 9$. Thus, to five decimal-place accuracy

$$
e \approx 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \approx 2.71828
$$

As a check, a calculator’s 12-digit representation of $e$ is $e \approx 2.71828182846$, which agrees with the preceding approximation when rounded to five decimal places. \(\triangleright\)

Example 8 Use the Remainder Estimation Theorem to find an interval containing $x = 0$ throughout which $f(x) = \cos x$ can be approximated by $p(x) = 1 - (x^2/2!)$ to three decimal-place accuracy.

Solution. We note first that $f(x) = \cos x$ has derivatives of all orders for every real number $x$, so the first hypothesis of the Remainder Estimation Theorem is satisfied over any interval that we choose. The given polynomial $p(x)$ is both the second and the third
9.7 Maclaurin and Taylor Polynomials

Maclaurin polynomial for \( \cos x \); we will choose the degree \( n \) of the polynomial to be as large as possible, so we will take \( n = 3 \). Our problem is to determine an interval on which the absolute value of the third remainder at \( x \) satisfies

\[ |R_3(x)| \leq 0.0005 \]

We will use the Remainder Estimation Theorem with \( f(x) = \cos x \), \( n = 3 \), and \( x_0 = 0 \). It follows from (14) that

\[ |R_3(x)| \leq \frac{M}{(3+1)!}|x-0|^{3+1} = \frac{M|x|^4}{24} \tag{17} \]

where \( M \) is an upper bound for \( |f^{(4)}(x)| = |\cos x| \). Since \( |\cos x| \leq 1 \) for every real number \( x \), we can take \( M = 1 \) in (17) to obtain

\[ |R_3(x)| \leq \frac{|x|^4}{24} \tag{18} \]

Thus we can achieve three decimal-place accuracy by choosing values of \( x \) for which

\[ \frac{|x|^4}{24} \leq 0.0005 \quad \text{or} \quad |x| \leq 0.3309 \]

so the interval \([-0.3309, 0.3309]\) is one option. We can check this answer by graphing \( |f(x) - p(x)| \) over the interval \([-0.3309, 0.3309]\) (Figure 9.7.7).

\[ \square \]

**QUICK CHECK EXERCISES 9.7** (See page 659 for answers.)

1. If \( f \) can be differentiated three times at \( 0 \), then the third Maclaurin polynomial for \( f \) is \( p_3(x) = \) __________.

2. The third Maclaurin polynomial for \( f(x) = e^{2x} \) is

\[ p_3(x) = \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \]

3. If \( f(2) = 3 \), \( f'(2) = -4 \), and \( f''(2) = 10 \), then the second Taylor polynomial for \( f \) about \( x = 2 \) is \( p_2(x) = \) __________.

4. The third Taylor polynomial for \( f(x) = x^3 \) about \( x = -1 \) is

\[ p_3(x) = \frac{f^{(0)}(-1)}{0!} + \frac{f^{(1)}(-1)}{1!}(x + 1) + \frac{f^{(2)}(-1)}{2!}(x + 1)^2 + \frac{f^{(3)}(-1)}{3!}(x + 1)^3 \]

**EXERCISE SET 9.7** [Graphing Utility]

1–2 In each part, find the local quadratic approximation of \( f \) at \( x = x_0 \), and use that approximation to find the local linear approximation of \( f \) at \( x_0 \). Use a graphing utility to graph \( f \) and the two approximations on the same screen.

1. (a) \( f(x) = e^{-x}; \ x_0 = 0 \)  
    (b) \( f(x) = \cos x; \ x_0 = 0 \)

2. (a) \( f(x) = \sin x; \ x_0 = \pi/2 \)  
    (b) \( f(x) = \sqrt{x}; \ x_0 = 1 \)

3. (a) Find the local quadratic approximation of \( \sqrt{x} \) at \( x_0 = 1 \).  
    (b) Use the result obtained in part (a) to approximate \( \sqrt{1.1} \), and compare the approximation to that produced directly by your calculating utility. [Note: See Example 1 of Section 3.5.]

4. (a) Find the local quadratic approximation of \( \cos x \) at \( x_0 = 0 \).  
    (b) Use the result obtained in part (a) to approximate \( \cos 2^\circ \), and compare the approximation to that produced directly by your calculating utility.

5. Use an appropriate local quadratic approximation to approximate \( \tan 61^\circ \), and compare the result to that produced directly by your calculating utility.

6. Use an appropriate local quadratic approximation to approximate \( \sqrt{36.03} \), and compare the result to that produced directly by your calculating utility.
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7–16 Find the Maclaurin polynomials of orders \( n = 0, 1, 2, 3, \) and 4, and then find the \( n \)th Maclaurin polynomials for the function in sigma notation.

7. \( e^{-x} \)  
8. \( e^{ax} \)  
9. \( \cos \pi x \)  
10. \( \sin \pi x \)  
11. \( \ln(1 + x) \)  
12. \( \frac{1}{1+x} \)  
13. \( \cosh x \)  
14. \( \sinh x \)  
15. \( x \sin x \)  
16. \( xe^x \)

17–24 Find the Taylor polynomials of orders \( n = 0, 1, 2, 3, \) and 4 about \( x = x_0 \), and then find the \( n \)th Taylor polynomial for the function in sigma notation.

17. \( e^x; \ x_0 = 1 \)  
18. \( e^{-x}; \ x_0 = \ln 2 \)  
19. \( \frac{1}{x}; \ x_0 = -1 \)  
20. \( \frac{1}{x+2}; \ x_0 = 3 \)  
21. \( \sin \pi x; \ x_0 = \frac{1}{2} \)  
22. \( \cos x; \ x_0 = \frac{\pi}{2} \)  
23. \( \ln x; \ x_0 = 1 \)  
24. \( \ln x; \ x_0 = e \)

25. (a) Find the third Maclaurin polynomial for \( f(x) = 1 + 2x - x^2 + x^3 \).

(b) Find the third Taylor polynomial about \( x = 1 \) for \( f(x) = 1 + 2(x-1)-(x-1)^2+(x-1)^3 \).

26. (a) Find the \( n \)th Maclaurin polynomial for \( f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n \).

(b) Find the \( n \)th Taylor polynomial about \( x = 1 \) for \( f(x) = c_0 + c_1(x-1) + c_2(x-1)^2 + \cdots + c_n(x-1)^n \).

27–30 Find the first four distinct Taylor polynomials about \( x = x_0 \), and use a graphing utility to graph the given function and the Taylor polynomials on the same screen.

27. \( f(x) = e^{-2x}; \ x_0 = 0 \)  
28. \( f(x) = \sin x; \ x_0 = \pi/2 \)  
29. \( f(x) = \cos x; \ x_0 = \pi \)  
30. \( \ln(x+1); \ x_0 = 0 \)

31–34 True–False Determine whether the statement is true or false. Explain your answer.

31. The equation of a tangent line to a differentiable function is a first-degree Taylor polynomial for that function.

32. The graph of a function \( f \) and the graph of its Maclaurin polynomial have a common \( y \)-intercept.

33. If \( p_0(x) \) is the sixth-degree Taylor polynomial for a function \( f \) about \( x = x_0 \), then \( p_0^{(6)}(x_0) = 4!f^{(6)}(x_0) \).

34. If \( p_4(x) \) is the fourth-degree Maclaurin polynomial for \( e^x \), then \( |e^x - p_4(2)| \leq \frac{9}{3!} \).

35–36 Use the method of Example 7 to approximate the given expression to the specified accuracy. Check your answer to that produced directly by your calculating utility.

35. \( \sqrt{e} \); four decimal-place accuracy

36. \( 1/e; \) three decimal-place accuracy

FOCUS ON CONCEPTS

37. Which of the functions graphed in the following figure is most likely to have \( p(x) = 1 - x + 2x^2 \) as its second-order Maclaurin polynomial? Explain your reasoning.

![Graphs of functions](image)

38. Suppose that the values of a function \( f \) and its first three derivatives at \( x = 1 \) are

\( f(1) = 2, \ f'(1) = -3, \ f''(1) = 0, \ f'''(1) = 6 \)

Find as many Taylor polynomials for \( f \) as you can about \( x = 1 \).

39. Let \( p_1(x) \) and \( p_2(x) \) be the local linear and local quadratic approximations of \( f(x) = e^{\sin x} \) at \( x = 0 \).

(a) Use a graphing utility to generate the graphs of \( f(x), p_1(x), \) and \( p_2(x) \) on the same screen for \(-1 \leq x \leq 1 \).

(b) Construct a table of values of \( f(x), p_1(x), \) and \( p_2(x) \) for \( x = -1.00, -0.75, -0.50, -0.25, 0, 0.25, 0.50, 0.75, 1.00 \). Round the values to three decimal places.

(c) Generate the graph of \( |f(x) - p_1(x)| \), and use the graph to determine an interval on which \( p_1(x) \) approximates \( f(x) \) with an error of at most \( \pm 0.01 \).

[Suggestion: Review the discussion relating to Figure 3.5.4.]

(d) Generate the graph of \( |f(x) - p_2(x)| \), and use the graph to determine an interval on which \( p_2(x) \) approximates \( f(x) \) with an error of at most \( \pm 0.01 \).

40. (a) The accompanying figure shows a sector of radius \( r \) and central angle \( 2\alpha \). Assuming that the angle \( \alpha \) is small, use the local quadratic approximation of \( \cos \alpha \) at \( \alpha = 0 \) to show that \( x \approx r\alpha^2/2 \).

(b) Assuming that the Earth is a sphere of radius 4000 mi, use the result in part (a) to approximate the maximum amount by which a 100 mi arc along the equator will diverge from its chord.

![Figure Ex-40](image)
41. (a) Find an interval \([0, b]\) over which \(e^x\) can be approximated by \(1 + x + \frac{x^2}{2!}\) to three decimal-place accuracy throughout the interval.

(b) Check your answer in part (a) by graphing \(|e^x - (1 + x + \frac{x^2}{2!})|\) over the interval you obtained.

42. Show that the \(n\)th Taylor polynomial for \(\sinh x\) about \(x = \ln 4\) is

\[
\sum_{k=0}^{n} \frac{16 - (-1)^k}{8k!} (x - \ln 4)^k
\]

43–46 Use the Remainder Estimation Theorem to find an interval containing \(x = 0\) over which \(f(x)\) can be approximated by \(p(x)\) to three decimal-place accuracy throughout the interval. Check your answer by graphing \(|f(x) - p(x)|\) over the interval you obtained.

43. \(f(x) = \sin x; \quad p(x) = x - \frac{x^3}{3!}\)

44. \(f(x) = \cos x; \quad p(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\)

45. \(f(x) = \frac{1}{1 + x}; \quad p(x) = 1 - x^2 + x^4\)

46. \(f(x) = \ln(1 + x); \quad p(x) = x - \frac{x^2}{2} + \frac{x^3}{3}\)

Quick Check Answers 9.7

1. \(f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3\)

2. 1; 2; \(\frac{4}{3}\)

3. 3 - 4(x - 2) + 5(x - 2)^2

4. -1; 5; -10; 10

5. (a) \(f(x) - p_n(x)\) (b) \(\frac{1}{2}|x - 2|^5\)

9.8 Maclaurin and Taylor Series; Power Series

Recall from the last section that the \(n\)th Taylor polynomial \(p_n(x)\) at \(x = x_0\) for a function \(f\) was defined so its value and the values of its first \(n\) derivatives match those of \(f\) at \(x_0\). This being the case, it is reasonable to expect that for values of \(x\) near \(x_0\) the values of \(p_n(x)\) will become better and better approximations of \(f(x)\) as \(n\) increases, and may possibly converge to \(f(x)\) as \(n \to +\infty\). We will explore this idea in this section.

MacLaurin and Taylor Series

In Section 9.7 we defined the \(n\)th Maclaurin polynomial for a function \(f\) as

\[
\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n
\]

and the \(n\)th Taylor polynomial for \(f\) about \(x = x_0\) as

\[
\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n
\]

It is not a big step to extend the notions of Maclaurin and Taylor polynomials to series by not stopping the summation index at \(n\). Thus, we have the following definition.
9.8.1 Definition If $f$ has derivatives of all orders at $x_0$, then we call the series
\[ \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \cdots \] (1)
the Taylor series for $f$ about $x = x_0$. In the special case where $x_0 = 0$, this series becomes
\[ \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(k)}(0)}{k!} x^k + \cdots \] (2)
in which case we call it the Maclaurin series for $f$.

Note that the $n$th Maclaurin and Taylor polynomials are the $n$th partial sums for the corresponding Maclaurin and Taylor series.

Example 1 Find the Maclaurin series for
(a) $e^x$  (b) $\sin x$  (c) $\cos x$  (d) $\frac{1}{1-x}$

Solution (a). In Example 2 of Section 9.7 we found that the $n$th Maclaurin polynomial for $e^x$ is
\[ p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \]
Thus, the Maclaurin series for $e^x$ is
\[ \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \]

Solution (b). In Example 3(a) of Section 9.7 we found that the Maclaurin polynomials for $\sin x$ are given by
\[ p_{2k+1}(x) = p_{2k+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (k = 0, 1, 2, \ldots) \]
Thus, the Maclaurin series for $\sin x$ is
\[ \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots \]

Solution (c). In Example 3(b) of Section 9.7 we found that the Maclaurin polynomials for $\cos x$ are given by
\[ p_{2k} = p_{2k+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} \quad (k = 0, 1, 2, \ldots) \]
Thus, the Maclaurin series for $\cos x$ is
\[ \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots \]
9.8 Maclaurin and Taylor Series; Power Series

**Solution (d).** In Example 5 of Section 9.7 we found that the $n$th Maclaurin polynomial for $1/(1 - x)$ is

$$p_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n \quad (n = 0, 1, 2, \ldots)$$

Thus, the Maclaurin series for $1/(1 - x)$ is

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

**Example 2** Find the Taylor series for $1/x$ about $x = 1$.

**Solution.** In Example 6 of Section 9.7 we found that the $n$th Taylor polynomial for $1/x$ about $x = 1$ is

$$\sum_{k=0}^{n} (-1)^k (x - 1)^k = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots + (-1)^n (x - 1)^n$$

Thus, the Taylor series for $1/x$ about $x = 1$ is

$$\sum_{k=0}^{\infty} (-1)^k (x - 1)^k = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots + (-1)^k (x - 1)^k + \cdots$$

**POWER SERIES IN $x$**

Maclaurin and Taylor series differ from the series that we have considered in Sections 9.3 to 9.6 in that their terms are not merely constants, but instead involve a variable. These are examples of power series, which we now define.

If $c_0, c_1, c_2, \ldots$ are constants and $x$ is a variable, then a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots$$

is called a power series in $x$. Some examples are

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

From Example 1, these are the Maclaurin series for the functions $1/(1 - x)$, $e^x$, and $\cos x$, respectively. Indeed, every Maclaurin series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(k)}(0)}{k!} x^k + \cdots$$

is a power series in $x$. 
RADIUS AND INTERVAL OF CONVERGENCE

If a numerical value is substituted for $x$ in a power series $\sum c_k x^k$, then the resulting series of numbers may either converge or diverge. This leads to the problem of determining the set of $x$-values for which a given power series converges; this is called its convergence set.

Observe that every power series in $x$ converges at $x = 0$, since substituting this value in (3) produces the series $c_0 + 0 + 0 + \cdots + 0 + \cdots$ whose sum is $c_0$. In some cases $x = 0$ may be the only number in the convergence set; in other cases the convergence set is some finite or infinite interval containing $x = 0$. This is the content of the following theorem, whose proof will be omitted.

**9.8.2 Theorem** For any power series in $x$, exactly one of the following is true:

(a) The series converges only for $x = 0$.

(b) The series converges absolutely (and hence converges) for all real values of $x$.

(c) The series converges absolutely (and hence converges) for all $x$ in some finite open interval $(-R, R)$ and diverges if $x < -R$ or $x > R$. At either of the values $x = R$ or $x = -R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

This theorem states that the convergence set for a power series in $x$ is always an interval centered at $x = 0$ (possibly just the value $x = 0$ itself or possibly infinite). For this reason, the convergence set of a power series in $x$ is called the interval of convergence. In the case where the convergence set is the single value $x = 0$ we say that the series has radius of convergence 0, in the case where the convergence set is $(−\infty, +\infty)$ we say that the series has radius of convergence $+\infty$, and in the case where the convergence set extends between $-R$ and $R$ we say that the series has radius of convergence $R$ (Figure 9.8.1).

![Figure 9.8.1](image)

**FINDING THE INTERVAL OF CONVERGENCE**

The usual procedure for finding the interval of convergence of a power series is to apply the ratio test for absolute convergence (Theorem 9.6.5). The following example illustrates how this works.

**Example 3** Find the interval of convergence and radius of convergence of the following power series.

(a) $\sum_{k=0}^{\infty} x^k$

(b) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

(c) $\sum_{k=0}^{\infty} k!x^k$

(d) $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k(k+1)}$
9.8 Maclaurin and Taylor Series; Power Series

**Solution (a).** Applying the ratio test for absolute convergence to the given series, we obtain
\[
\rho = \lim_{k \to +\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to +\infty} \frac{|x|^{k+1}}{|x|^k} = |x|
\]
so the series converges absolutely if \( \rho = |x| < 1 \) and diverges if \( \rho = |x| > 1 \). The test is inconclusive if \( |x| = 1 \) (i.e., if \( x = 1 \) or \( x = -1 \)), which means that we will have to investigate convergence at these values separately. At these values the series becomes
\[
\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + 1 + \cdots \quad \text{for} \quad x = 1
\]
\[
\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - \cdots \quad \text{for} \quad x = -1
\]
both of which diverge; thus, the interval of convergence for the given power series is \((-1, 1)\), and the radius of convergence is \(R = 1\).

**Solution (b).** Applying the ratio test for absolute convergence to the given series, we obtain
\[
\rho = \lim_{k \to +\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to +\infty} \frac{|x|^{k+1}}{(k+1)!} \cdot \frac{k!}{|x|^k} = \lim_{k \to +\infty} \frac{|x|}{k+1} = 0
\]
Since \( \rho < 1 \) for all \( x \), the series converges absolutely for all \( x \). Thus, the interval of convergence is \((-\infty, +\infty)\) and the radius of convergence is \( R = +\infty \).

**Solution (c).** If \( x \neq 0 \), then the ratio test for absolute convergence yields
\[
\rho = \lim_{k \to +\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to +\infty} \frac{1}{k+1} = +\infty
\]
Therefore, the series diverges for all nonzero values of \( x \). Thus, the interval of convergence is the single value \( x = 0 \) and the radius of convergence is \( R = 0 \).

**Solution (d).** Since \( |(-1)^k| = |(-1)^{k+1}| = 1 \), we obtain
\[
\rho = \lim_{k \to +\infty} \frac{|x|}{k+1} = \frac{|x|}{k+1} \quad \text{for all} \quad x
\]
The ratio test for absolute convergence implies that the series converges absolutely if \( |x| < 3 \) and diverges if \( |x| > 3 \). The ratio test fails to provide any information when \( |x| = 3 \), so the cases \( x = -3 \) and \( x = 3 \) need separate analyses. Substituting \( x = -3 \) in the given series yields
\[
\sum_{k=0}^{\infty} \frac{(-1)^k(-3)^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{1}{k+1}
\]
which is the divergent harmonic series \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \). Substituting \( x = 3 \) in the given series yields
\[
\sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots
\]
which is the conditionally convergent alternating harmonic series. Thus, the interval of convergence for the given series is \((-3, 3]\) and the radius of convergence is \( R = 3 \). ☑
Chapter 9 / Infinite Series

POWER SERIES IN $x - x_0$

If $x_0$ is a constant, and if $x$ is replaced by $x - x_0$ in (3), then the resulting series has the form

$$
\sum_{k=0}^{\infty} c_k(x - x_0)^k = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_k(x - x_0)^k + \cdots
$$

This is called a power series in $x - x_0$. Some examples are

$$
\begin{align*}
\sum_{k=0}^{\infty} \frac{(x - 1)^k}{k+1} &= 1 + \frac{(x - 1)}{2} + \frac{(x - 1)^2}{3} + \frac{(x - 1)^3}{4} + \cdots \quad x_0 = 1 \\
\sum_{k=0}^{\infty} \frac{(-1)^k(x + 3)^k}{k!} &= 1 - (x + 3) + \frac{(x + 3)^2}{2!} - \frac{(x + 3)^3}{3!} + \cdots \quad x_0 = -3
\end{align*}
$$

The first of these is a power series in $x - 1$ and the second is a power series in $x + 3$. Note that a power series in $x$ is a power series in $x - x_0$ in which $x_0 = 0$. More generally, the Taylor series

$$
\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k
$$

is a power series in $x - x_0$.

The main result on convergence of a power series in $x - x_0$ can be obtained by substituting $x - x_0$ for $x$ in Theorem 9.8.2. This leads to the following theorem.

9.8.3 THEOREM For a power series $\sum c_k(x - x_0)^k$, exactly one of the following statements is true:

(a) The series converges only for $x = x_0$.

(b) The series converges absolutely (and hence converges) for all real values of $x$.

(c) The series converges absolutely (and hence converges) for all $x$ in some finite open interval $(x_0 - R, x_0 + R)$ and diverges if $x < x_0 - R$ or $x > x_0 + R$. At either of the values $x = x_0 - R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

It follows from this theorem that the set of values for which a power series in $x - x_0$ converges is always an interval centered at $x = x_0$; we call this the interval of convergence (Figure 9.8.2). In part (a) of Theorem 9.8.3 the interval of convergence reduces to the single value $x = x_0$, in which case we say that the series has radius of convergence $R = 0$; in part
(b) the interval of convergence is infinite (the entire real line), in which case we say that the series has \textit{radius of convergence} \( R = +\infty \); and in part (c) the interval extends between \( x_0 - R \) and \( x_0 + R \), in which case we say that the series has \textit{radius of convergence} \( R \).

\section*{Example 4} Find the interval of convergence and radius of convergence of the series

\[ \sum_{k=1}^{\infty} \frac{(x - 5)^k}{k^2} \]

\textbf{Solution.} We apply the ratio test for absolute convergence.

\[ \rho = \lim_{k \to +\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to +\infty} \left| \frac{(x - 5)^{k+1}}{(k + 1)^2} \cdot \frac{k^2}{(x - 5)^k} \right| \]

\[ = \lim_{k \to +\infty} \left| -5 \right| \left( \frac{k}{k + 1} \right)^2 \]

\[ = \left| x - 5 \right| \lim_{k \to +\infty} \left( \frac{1 + 1/k}{1} \right)^2 = \left| x - 5 \right| \]

Thus, the series converges absolutely if \( \left| x - 5 \right| < 1 \), or \(-1 < x - 5 < 1\), or \(4 < x < 6\).

The series diverges if \( x < 4 \) or \( x > 6 \).

To determine the convergence behavior at the endpoints \( x = 4 \) and \( x = 6 \), we substitute these values in the given series. If \( x = 6 \), the series becomes

\[ \sum_{k=1}^{\infty} \frac{1^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \]

which is a convergent \( p \)-series (\( p = 2 \)). If \( x = 4 \), the series becomes

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots \]

Since this series converges absolutely, the interval of convergence for the given series is \([4, 6]\). The radius of convergence is \( R = 1 \) (Figure 9.8.3).

\begin{center}
\begin{tabular}{ccc}
Series diverges & Series converges absolutely & Series diverges \\
\hline
4 & R = 1 & 6 \\
\hline
x_0 = 5 & R = 1 & \hline
\end{tabular}
\end{center}

\section*{FUNCTIONS DEFINED BY POWER SERIES}

If a function \( f \) is expressed as a power series on some interval, then we say that the power series \textit{represents} \( f \) on that interval. For example, we saw in Example 4(a) of Section 9.3 that

\[ \frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k \]

if \( |x| < 1 \), so this power series represents the function \( 1/(1 - x) \) on the interval \(-1 < x < 1\).
TECHNOLOGY MASTERY

Many computer algebra systems have the Bessel functions as part of their libraries. If you have a CAS with Bessel functions, use it to generate the graphs in Figure 9.8.4.

QUICK CHECK EXERCISES 9.8  (See page 668 for answers.)

1. If \( f \) has derivatives of all orders at \( x_0 \), then the Taylor series for \( f \) about \( x = x_0 \) is defined to be

   \[
   \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k
   \]

2. Since

   \[
   \lim_{k \to +\infty} \frac{2^{k+1}x^{k+1}}{2^k x^k} = 2|x|
   \]

   the radius of convergence for the infinite series \( \sum_{k=0}^{\infty} 2^k x^k \) is ________.

3. Since

   \[
   \lim_{k \to +\infty} \left| \frac{(3^{k+1}x^{k+1})(k+1)!}{(3^k x^k)k!} \right| = \lim_{k \to +\infty} \left| \frac{3x}{k+1} \right| = 0
   \]

   the interval of convergence for the series \( \sum_{k=0}^{\infty} (3^k/k!)x^k \) is ________.

4. (a) Since

   \[
   \lim_{k \to +\infty} \frac{(x-4)^{k+1}/\sqrt{k+1}}{(x-4)^k/\sqrt{k}} = \lim_{k \to +\infty} \left| \frac{k}{\sqrt{k+1}}(x-4) \right| = |x-4|
   \]

   the radius of convergence for the infinite series \( \sum_{k=1}^{\infty} (1/\sqrt{k})(x-4)^k \) is ________.

   (b) When \( x = 3 \),

   \[
   \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}(x-4)^k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}(-1)^k
   \]

   Does this series converge or diverge?

   (c) When \( x = 5 \),

   \[
   \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}(x-4)^k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}
   \]

   Does this series converge or diverge?

   (d) The interval of convergence for the infinite series \( \sum_{k=1}^{\infty} (1/\sqrt{k})(x-4)^k \) is ________.
EXERCISE SET 9.8

1–10 Use sigma notation to write the Maclaurin series for the function.

1. \( e^{-x} \)
2. \( e^{ax} \)
3. \( \cos \pi x \)
4. \( \sin \pi x \)
5. \( \ln(1 + x) \)
6. \( \frac{1}{1 + x} \)
7. \( \cosh x \)
8. \( \sinh x \)
9. \( x \sin x \)
10. \( x e^x \)

11–18 Use sigma notation to write the Taylor series about \( x = x_0 \) for the function.

11. \( e^x; \ x_0 = 1 \)
12. \( e^{-x}; \ x_0 = \ln 2 \)
13. \( \frac{1}{x}; \ x_0 = -1 \)
14. \( \frac{1}{x + 2}; \ x_0 = 3 \)
15. \( \sin \pi x; \ x_0 = \frac{1}{2} \)
16. \( \cos x; \ x_0 = \frac{\pi}{2} \)
17. \( \ln x; \ x_0 = 1 \)
18. \( \ln x; \ x_0 = e \)

19–22 Find the interval of convergence of the power series, and find a familiar function that is represented by the power series on that interval.

19. \( 1 - x + x^2 - x^3 + \ldots + (-1)^j x^j + \ldots \)
20. \( 1 + x^2 + x^4 + \ldots + x^{2k} + \ldots \)
21. \( 1 + (x - 2) + (x - 2)^2 + \ldots + (x - 2)^j + \ldots \)
22. \( 1 - (x + 3) + (x + 3)^2 - (x + 3)^3 + \ldots + (-1)^j (x + 3)^j \ldots \)

23. Suppose that the function \( f \) is represented by the power series

\[ f(x) = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots + (-1)^j \frac{x^j}{2^j} + \ldots \]

(a) Find the domain of \( f \). (b) Find \( f(0) \) and \( f(1) \).

24. Suppose that the function \( f \) is represented by the power series

\[ f(x) = 1 - \frac{x - 5}{3} + \frac{(x - 5)^2}{3^2} - \frac{(x - 5)^3}{3^3} + \ldots \]

(a) Find the domain of \( f \). (b) Find \( f(3) \) and \( f(6) \).

25–28 True–False Determine whether the statement is true or false. Explain your answer.

25. If a power series in \( x \) converges conditionally at \( x = 3 \), then the series converges if \( |x| < 3 \) and diverges if \( |x| > 3 \).
26. The ratio test is often useful to determine convergence at the endpoints of the interval of convergence of a power series.
27. The Maclaurin series for a polynomial function has radius of convergence \( +\infty \).
28. The series \( \sum_{k=0}^{\infty} \frac{x^k}{k!} \) converges if \( |x| < 1 \).
29–50 Find the radius of convergence and the interval of convergence.

51. Use the root test to find the interval of convergence of

\[ \sum_{k=0}^{\infty} \frac{x^k}{(\ln k)^k} \]

52. Find the domain of the function

\[ f(x) = \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k - 1) x^k}{(2k - 2)!} \]

53. Show that the series

\[ 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots \]

is the Maclaurin series for the function

\[ f(x) = \begin{cases} \cos \sqrt{x}, & x \geq 0 \\ \cosh \sqrt{-x}, & x < 0 \end{cases} \]

[Hint: Use the Maclaurin series for \( \cos x \) and \( \cosh x \) to obtain series for \( \cos \sqrt{x} \), where \( x \geq 0 \), and \( \cosh \sqrt{-x} \), where \( x < 0 \).]

54. If a function \( f \) is represented by a power series on an interval, then the graphs of the partial sums can be used as approximations to the graph of \( f \).

(a) Use a graphing utility to generate the graph of \( 1/(1-x) \) together with the graphs of the first four partial sums of its Maclaurin series over the interval \((-1, 1)\).
55. Prove:
(a) If \( f \) is an even function, then all odd powers of \( x \) in its Maclaurin series have coefficient 0.
(b) If \( f \) is an odd function, then all even powers of \( x \) in its Maclaurin series have coefficient 0.

56. Suppose that the power series \( \sum c_k(x - x_0)^k \) has radius of convergence \( R \) and \( p \) is a nonzero constant. What can you say about the radius of convergence of the power series \( \sum pc_k(x - x_0)^k \)? Explain your reasoning. [Hint: See Theorem 9.4.3.]

57. Suppose that the power series \( \sum c_k(x - x_0)^k \) has a finite radius of convergence \( R \), and the power series \( \sum d_k(x - x_0)^k \) has a radius of convergence of \( +\infty \). What can you say about the radius of convergence of \( \sum(c_k + d_k)(x - x_0)^k \)? Explain your reasoning.

58. Suppose that the power series \( \sum c_k(x - x_0)^k \) has a finite radius of convergence \( R_1 \) and the power series \( \sum d_k(x - x_0)^k \) has a finite radius of convergence \( R_2 \). What can you say about the radius of convergence of \( \sum(c_k + d_k)(x - x_0)^k \)? Explain your reasoning. [Hint: The case \( R_1 = R_2 \) requires special attention.]

59. Show that if \( p \) is a positive integer, then the power series \[ \sum_{k=0}^{\infty} \frac{(pk)!}{(k!)^p} x^k \] has a radius of convergence of \( 1/p^p \).

60. Show that if \( p \) and \( q \) are positive integers, then the power series \[ \sum_{k=0}^{\infty} \frac{(k + p)!}{k!(k + q)!} x^k \] has a radius of convergence of \( +\infty \).

61. Show that the power series representation of the Bessel function \( J_1(x) \) converges for all \( x \) [Formula (5)].

62. Approximate the values of the Bessel functions \( J_0(x) \) and \( J_1(x) \) at \( x = 1 \), each to four decimal-place accuracy.

63. If the constant \( p \) in the general \( p \)-series is replaced by a variable \( x \) for \( x > 1 \), then the resulting function is called the Riemann zeta function and is denoted by \[ \zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x} \]
(a) Let \( s_n \) be the \( n \)th partial sum of the series for \( \zeta(3.7) \). Find \( n \) such that \( s_n \) approximates \( \zeta(3.7) \) to two decimal-place accuracy, and calculate \( s_n \) using this value of \( n \). [Hint: Use the right inequality in Exercise 36(b) of Section 9.4 with \( f(x) = 1/x^{3.7} \].
(b) Determine whether your CAS can evaluate the Riemann zeta function directly. If so, compare the value produced by the CAS to the value of \( s_n \) obtained in part (a).

64. Prove: If \( \lim_{x \to +\infty} |c_k|^{1/k} = L \), where \( L \neq 0 \), then \( 1/L \) is the radius of convergence of the power series \( \sum_{k=0}^{\infty} c_k x^k \).

65. Prove: If the power series \( \sum_{k=0}^{\infty} c_k x^k \) has radius of convergence \( R \), then the series \( \sum_{k=0}^{\infty} c_k x^{2k} \) has radius of convergence \( \sqrt{R} \).

66. Prove: If the interval of convergence of the series \( \sum_{k=0}^{\infty} c_k(x - x_0)^k \) is \( (x_0 - R, x_0 + R) \), then the series converges conditionally at \( x_0 + R \).

67. Writing The sine function can be defined geometrically from the unit circle or analytically from its Maclaurin series. Discuss the advantages of each representation with regard to providing information about the sine function.

\[ \text{\textbullet \quad \text{QUICK CHECK ANSWERS 9.8}} \]

1. \( \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \) \quad 2. \( \frac{1}{2} \) \quad 3. \( (-\infty, +\infty) \) \quad 4. (a) 1 (b) converges (c) diverges (d) [3, 5]

\[ \text{\textbullet \quad \text{9.9 CONVERGENCE OF TAYLOR SERIES}} \]

In this section we will investigate when a Taylor series for a function converges to that function on some interval, and we will consider how Taylor series can be used to approximate values of trigonometric, exponential, and logarithmic functions.

\[ \text{\textbullet \quad \text{THE CONVERGENCE PROBLEM FOR TAYLOR SERIES}} \]

Recall that the \( n \)th Taylor polynomial for a function \( f \) about \( x = x_0 \) has the property that its value and the values of its first \( n \) derivatives match those of \( f \) at \( x_0 \). As \( n \) increases,
more and more derivatives match up, so it is reasonable to hope that for values of $x$ near $x_0$ the values of the Taylor polynomials might converge to the value of $f(x)$; that is,

$$f(x) = \lim_{n \to +\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

(1)

However, the $n$th Taylor polynomial for $f$ is the $n$th partial sum of the Taylor series for $f$, so (1) is equivalent to stating that the Taylor series for $f$ converges at $x$, and its sum is $f(x)$. Thus, we are led to consider the following problem.

9.9.1 Problem Given a function $f$ that has derivatives of all orders at $x = x_0$, determine whether there is an open interval containing $x_0$ such that $f(x)$ is the sum of its Taylor series about $x = x_0$ at each point in the interval; that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

(2)

for all values of $x$ in the interval.

One way to show that (1) holds is to show that

$$\lim_{n \to +\infty} \left[ f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \right] = 0$$

However, the difference appearing on the left side of this equation is the $n$th remainder for the Taylor series [Formula (12) of Section 9.7]. Thus, we have the following result.

9.9.2 Theorem The equality

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

holds at a point $x$ if and only if $\lim_{n \to +\infty} R_n(x) = 0$.

ESTIMATING THE $n$TH REMAINDER

It is relatively rare that one can prove directly that $R_n(x) \to 0$ as $n \to +\infty$. Usually, this is proved indirectly by finding appropriate bounds on $|R_n(x)|$ and applying the Squeezing Theorem for Sequences. The Remainder Estimation Theorem (Theorem 9.7.4) provides a useful bound for this purpose. Recall that this theorem asserts that if $M$ is an upper bound for $|f^{(n+1)}(x)|$ on an interval containing $x_0$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1}$$

(3)

for all $x$ in that interval.

The following example illustrates how the Remainder Estimation Theorem is applied.

Example 1 Show that the Maclaurin series for $\cos x$ converges to $\cos x$ for all $x$; that is,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (-\infty < x < +\infty)$$
Solution. From Theorem 9.9.2 we must show that \( R_n(x) \to 0 \) for all \( x \) as \( n \to +\infty \). For this purpose let \( f(x) = \cos x \), so that for all \( x \) we have
\[
f^{(n+1)}(x) = \pm \cos x \quad \text{or} \quad f^{(n+1)}(x) = \pm \sin x
\]
In all cases we have \( |f^{(n+1)}(x)| \leq 1 \), so we can apply (3) with \( M = 1 \) and \( x_0 = 0 \) to conclude that
\[
0 \leq |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad (4)
\]
However, it follows from Formula (5) of Section 9.2 with \( n+1 \) in place of \( n \) and \( |x| \) in place of \( x \) that
\[
\lim_{n \to +\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \quad (5)
\]
Using this result and the Squeezing Theorem for Sequences (Theorem 9.1.5), it follows from (4) that \( |R_n(x)| \to 0 \) and hence that \( R_n(x) \to 0 \) as \( n \to +\infty \) (Theorem 9.1.6). Since this is true for all \( x \), we have proved that the Maclaurin series for \( \cos x \) converges to \( \cos x \) for all \( x \). This is illustrated in Figure 9.9.1, where we can see how successive partial sums approximate the cosine curve more and more closely.

The method of Example 1 can be easily modified to prove that the Taylor series for \( \sin x \) and \( \cos x \) about any point \( x = x_0 \) converge to \( \sin x \) and \( \cos x \), respectively, for all \( x \) (Exercises 21 and 22). For reference, some of the most important Maclaurin series are listed in Table 9.9.1 at the end of this section.

\section*{APPROXIMATING TRIGONOMETRIC FUNCTIONS}

In general, to approximate the value of a function \( f \) at a point \( x \) using a Taylor series, there are two basic questions that must be answered:

- About what point \( x_0 \) should the Taylor series be expanded?
- How many terms in the series should be used to achieve the desired accuracy?

In response to the first question, \( x_0 \) needs to be a point at which the derivatives of \( f \) can be evaluated easily, since these values are needed for the coefficients in the Taylor series. Furthermore, if the function \( f \) is being evaluated at \( x \), then \( x_0 \) should be chosen as close as possible to \( x \), since Taylor series tend to converge more rapidly near \( x_0 \). For example, to approximate \( \sin 3^\circ = \pi/60 \) radians), it would be reasonable to take \( x_0 = 0 \), since \( \pi/60 \) is close to 0 and the derivatives of \( \sin x \) are easy to evaluate at 0. On the other hand, to approximate \( \sin 85^\circ = 17\pi/36 \) radians), it would be more natural to take \( x_0 = \pi/2 \), since \( 17\pi/36 \) is close to \( \pi/2 \) and the derivatives of \( \sin x \) are easy to evaluate at \( \pi/2 \).

In response to the second question posed above, the number of terms required to achieve a specific accuracy needs to be determined on a problem-by-problem basis. The next example gives two methods for doing this.

\begin{example}
Use the Maclaurin series for \( \sin x \) to approximate \( \sin 3^\circ \) to five decimal-place accuracy.
\end{example}

\[ y = \cos x \]

\[ p_{2n} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} \]
9.9 Convergence of Taylor Series

**Solution.** In the Maclaurin series
\[
\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{6}
\]
the angle \(x\) is assumed to be in radians (because the differentiation formulas for the trigonometric functions were derived with this assumption). Since \(3\degree = \pi/60\) radians, it follows from (6) that
\[
\sin 3\degree = \sin \frac{\pi}{60} = \left( \frac{\pi}{60} \right) - \frac{(\pi/60)^3}{3!} + \frac{(\pi/60)^5}{5!} - \frac{(\pi/60)^7}{7!} + \cdots \tag{7}
\]
We must now determine how many terms in the series are required to achieve five decimal-place accuracy. We will consider two possible approaches, one using the Remainder Estimation Theorem (Theorem 9.7.4) and the other using the fact that (7) satisfies the hypotheses of the alternating series test (Theorem 9.6.1).

**Method 1. (The Remainder Estimation Theorem)**
Since we want to achieve five decimal-place accuracy, our goal is to choose \(n\) so that the absolute value of the \(n\)th remainder at \(x = \pi/60\) does not exceed \(0.000005 = 5 \times 10^{-6}\); that is,
\[
|R_n \left( \frac{\pi}{60} \right) | \leq 0.000005 \tag{8}
\]
However, if we let \(f(x) = \sin x\), then \(f^{(n+1)}(x)\) is either \(\pm \sin x\) or \(\pm \cos x\), and in either case \(|f^{(n+1)}(x)| \leq 1\) for all \(x\). Thus, it follows from the Remainder Estimation Theorem with \(M = 1, x_0 = 0\), and \(x = \pi/60\) that
\[
|R_n \left( \frac{\pi}{60} \right) | \leq \frac{(\pi/60)^{n+1}}{(n+1)!}
\]
Thus, we can satisfy (8) by choosing \(n\) so that
\[
\frac{(\pi/60)^{n+1}}{(n+1)!} \leq 0.000005
\]
With the help of a calculating utility you can verify that the smallest value of \(n\) that meets this criterion is \(n = 3\). Thus, to achieve five decimal-place accuracy we need only keep terms up to the third power in (7). This yields
\[
\sin 3\degree \approx \left( \frac{\pi}{60} \right) - \frac{(\pi/60)^3}{3!} \approx 0.05234 \tag{9}
\]
(verify). As a check, a calculator gives \(\sin 3\degree \approx 0.05233595624\), which agrees with (9) when rounded to five decimal places.

**Method 2. (The Alternating Series Test)**
We leave it for you to check that (7) satisfies the hypotheses of the alternating series test (Theorem 9.6.1).

Let \(s_n\) denote the sum of the terms in (7) up to and including the \(n\)th power of \(\pi/60\). Since the exponents in the series are odd integers, the integer \(n\) must be odd, and the exponent of the first term not included in the sum \(s_n\) must be \(n + 2\). Thus, it follows from part (b) of Theorem 9.6.2 that
\[
|\sin 3\degree - s_n| < \frac{(\pi/60)^{n+2}}{(n+2)!}
\]
This means that for five decimal-place accuracy we must look for the first positive odd integer \(n\) such that
\[
\frac{(\pi/60)^{n+2}}{(n+2)!} \leq 0.000005
\]
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With the help of a calculating utility you can verify that the smallest value of \( n \) that meets this criterion is \( n = 3 \). This agrees with the result obtained above using the Remainder Estimation Theorem and hence leads to approximation (9) as before.

ROUND OFF AND TRUNCATION ERROR

There are two types of errors that occur when computing with series. The first, called truncation error, is the error that results when a series is approximated by a partial sum; and the second, called roundoff error, is the error that arises from approximations in numerical computations. For example, in our derivation of (9) we took \( n = 3 \) to keep the truncation error below 0.000005. However, to evaluate the partial sum we had to approximate \( \pi \), thereby introducing roundoff error. Had we not exercised some care in choosing this approximation, the roundoff error could easily have degraded the final result.

Methods for estimating and controlling roundoff error are studied in a branch of mathematics called numerical analysis. However, as a rule of thumb, to achieve \( n \) decimal-place accuracy in a final result, all intermediate calculations must be accurate to at least \( n + 1 \) decimal places. Thus, in (9) at least six decimal-place accuracy in \( \pi \) is required to achieve the five decimal-place accuracy in the final numerical result. As a practical matter, a good working procedure is to perform all intermediate computations with the maximum number of digits that your calculating utility can handle and then round at the end.

APPROXIMATING EXPONENTIAL FUNCTIONS

Example 3 Show that the Maclaurin series for \( e^x \) converges to \( e^x \) for all \( x \); that is,

\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \quad (-\infty < x < +\infty)
\]

Solution. Let \( f(x) = e^x \), so that

\[
f^{(n+1)}(x) = e^x
\]

We want to show that \( R_n(x) \to 0 \) as \( n \to +\infty \) for all \( x \) in the interval \( -\infty < x < +\infty \). However, it will be helpful here to consider the cases \( x \leq 0 \) and \( x > 0 \) separately. If \( x \leq 0 \), then we will take the interval in the Remainder Estimation Theorem (Theorem 9.7.4) to be \([x, 0]\), and if \( x > 0 \), then we will take it to be \([0, x]\). Since \( f^{(n+1)}(x) = e^x \) is an increasing function, it follows that if \( c \) is in the interval \([x, 0]\), then

\[
|f^{(n+1)}(c)| \leq |f^{(n+1)}(0)| = e^0 = 1
\]

and if \( c \) is in the interval \([0, x]\), then

\[
|f^{(n+1)}(c)| \leq |f^{(n+1)}(x)| = e^x
\]

Thus, we can apply Theorem 9.7.4 with \( M = 1 \) in the case where \( x \leq 0 \) and with \( M = e^x \) in the case where \( x > 0 \). This yields

\[
0 \leq |R_n(x)| \leq |x|^{n+1} \quad \frac{1}{(n + 1)!} \quad \text{if} \quad x \leq 0
\]

\[
0 \leq |R_n(x)| \leq e^x \quad \frac{|x|^{n+1}}{(n + 1)!} \quad \text{if} \quad x > 0
\]

Thus, in both cases it follows from (5) and the Squeezing Theorem for Sequences that \( |R_n(x)| \to 0 \) as \( n \to +\infty \), which in turn implies that \( R_n(x) \to 0 \) as \( n \to +\infty \). Since this is true for all \( x \), we have proved that the Maclaurin series for \( e^x \) converges to \( e^x \) for all \( x \). ◇
9.9 Convergence of Taylor Series

Since the Maclaurin series for $e^x$ converges to $e^x$ for all $x$, we can use partial sums of the Maclaurin series to approximate powers of $e$ to arbitrary precision. Recall that in Example 7 of Section 9.7 we were able to use the Remainder Estimation Theorem to determine that evaluating the ninth Maclaurin polynomial for $e^x$ at $x = 1$ yields an approximation for $e$ with five decimal-place accuracy:

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \approx 2.71828$$

### APPROXIMATING LOGARITHMS

The Maclaurin series

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (-1 < x \leq 1) \tag{10}$$

is the starting point for the approximation of natural logarithms. Unfortunately, the usefulness of this series is limited because of its slow convergence and the restriction $-1 < x \leq 1$. However, if we replace $x$ by $-x$ in this series, we obtain

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad (-1 \leq x < 1) \tag{11}$$

and on subtracting (11) from (10) we obtain

$$\ln \left( \frac{1 + x}{1 - x} \right) = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \right) \quad (-1 < x < 1) \tag{12}$$

Series (12), first obtained by James Gregory in 1668, can be used to compute the natural logarithm of any positive number $y$ by letting

$$y = \frac{1 + x}{1 - x}$$

or, equivalently,

$$x = \frac{y - 1}{y + 1} \tag{13}$$

and noting that $-1 < x < 1$. For example, to compute $\ln 2$ we let $y = 2$ in (13), which yields $x = \frac{1}{3}$. Substituting this value in (12) gives

$$\ln 2 \approx 2 \left[ \frac{1}{3} + \left( \frac{1}{3} \right)^3 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \right] \tag{14}$$

In Exercise 19 we will ask you to show that five decimal-place accuracy can be achieved using the partial sum with terms up to and including the 13th power of $\frac{1}{3}$. Thus, to five decimal-place accuracy

$$\ln 2 \approx 2 \left[ \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{13} \right] \approx 0.69315$$

(verify). As a check, a calculator gives $\ln 2 \approx 0.69314718056$, which agrees with the preceding approximation when rounded to five decimal places.

### APPROXIMATING $\pi$

In the next section we will show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (-1 \leq x \leq 1) \tag{15}$$

Letting $x = 1$, we obtain

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$
or

\[ f(x) = \left[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right] \]

This famous series, obtained by Leibniz in 1674, converges too slowly to be of computational value. A more practical procedure for approximating \( \pi \) uses the identity

\[ \frac{\pi}{4} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} \]

which was derived in Exercise 60 of Section 0.4. By using this identity and series (15) to approximate \( \tan^{-1} \frac{1}{2} \) and \( \tan^{-1} \frac{1}{3} \), the value of \( \pi \) can be approximated efficiently to any degree of accuracy.

### BINOMIAL SERIES

If \( m \) is a real number, then the Maclaurin series for \((1 + x)^m\) is called the **binomial series**; it is given by

\[
1 + mx + \frac{m(m - 1)}{2!} x^2 + \frac{m(m - 1)(m - 2)}{3!} x^3 + \cdots + \frac{m(m - 1) \cdots (m - k + 1)}{k!} x^k + \cdots
\]

In the case where \( m \) is a nonnegative integer, the function \( f(x) = (1 + x)^m \) is a polynomial of degree \( m \), so

\[
f^{(m+1)}(0) = f^{(m+2)}(0) = f^{(m+3)}(0) = \cdots = 0
\]

and the binomial series reduces to the familiar binomial expansion

\[
(1 + x)^m = 1 + mx + \frac{m(m - 1)}{2!} x^2 + \frac{m(m - 1)(m - 2)}{3!} x^3 + \cdots + x^m
\]

which is valid for \( -1 < x < +\infty \).

It can be proved that if \( m \) is not a nonnegative integer, then the binomial series converges to \((1 + x)^m\) if \( |x| < 1 \). Thus, for such values of \( x \)

\[
(1 + x)^m = 1 + mx + \frac{m(m - 1)}{2!} x^2 + \cdots + \frac{m(m - 1) \cdots (m - k + 1)}{k!} x^k + \cdots
\]

or in sigma notation,

\[
(1 + x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m - 1) \cdots (m - k + 1)}{k!} x^k \quad \text{if } |x| < 1
\]

#### Example 4

Find binomial series for

(a) \( \frac{1}{(1 + x)^2} \)

(b) \( \frac{1}{\sqrt{1 + x}} \)

**Solution (a).** Since the general term of the binomial series is complicated, you may find it helpful to write out some of the beginning terms of the series, as in Formula (17), to see developing patterns. Substituting \( m = -2 \) in this formula yields

\[
\frac{1}{(1 + x)^2} = (1 + x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!} x^2 + \cdots
\]

\[
= 1 - 2x + \frac{3!}{2!} x^2 - \frac{4!}{3!} x^3 + \frac{5!}{4!} x^4 + \cdots
\]

\[
= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots
\]

\[
= \sum_{k=0}^{\infty} (-1)^k (k + 1) x^k
\]
Solution (b). Substituting \( m = -\frac{1}{2} \) in (17) yields

\[
\frac{1}{\sqrt{1 + x}} = 1 - \frac{1}{2} x + \frac{(-\frac{1}{2})(-\frac{1}{2} - 1)}{2!} x^2 + \frac{(-\frac{1}{2})(-\frac{1}{2} - 1)(-\frac{1}{2} - 2)}{3!} x^3 + \ldots
\]

\[
= 1 - \frac{1}{2} x + \frac{1 \cdot 3}{2^2 \cdot 2!} x^2 - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!} x^3 + \ldots
\]

\[
= 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \ldots (2k - 1)}{2^k k!} x^k
\]

Figure 9.9.2 shows the graphs of the functions in Example 4 compared to their third-degree Maclaurin polynomials.

### SOME IMPORTANT MACLAURIN SERIES

For reference, Table 9.9.1 lists the Maclaurin series for some of the most important functions, together with a specification of the intervals over which the Maclaurin series converge to those functions. Some of these results are derived in the exercises and others will be derived in the next section using some special techniques that we will develop.

<table>
<thead>
<tr>
<th>MACLAURIN SERIES</th>
<th>INTERVAL OF CONVERGENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k )</td>
<td>( -1 &lt; x &lt; 1 )</td>
</tr>
<tr>
<td>( \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} )</td>
<td>( -1 &lt; x &lt; 1 )</td>
</tr>
<tr>
<td>( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} )</td>
<td>( -\infty &lt; x &lt; +\infty )</td>
</tr>
<tr>
<td>( \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} )</td>
<td>( -\infty &lt; x &lt; +\infty )</td>
</tr>
<tr>
<td>( \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} )</td>
<td>( -\infty &lt; x &lt; +\infty )</td>
</tr>
<tr>
<td>( \ln (1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} )</td>
<td>( -1 &lt; x \leq 1 )</td>
</tr>
<tr>
<td>( \tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k + 1} )</td>
<td>( -1 \leq x \leq 1 )</td>
</tr>
<tr>
<td>( \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} )</td>
<td>( -\infty &lt; x &lt; +\infty )</td>
</tr>
<tr>
<td>( \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} )</td>
<td>( -\infty &lt; x &lt; +\infty )</td>
</tr>
<tr>
<td>( (1 + x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\ldots(m-k+1)}{k!} x^k )</td>
<td>( -1 &lt; x &lt; 1 ) ( m \neq 0, 1, 2, \ldots )</td>
</tr>
</tbody>
</table>

The behavior at the endpoints depends on \( m \): For \( m > 0 \) the series converges absolutely at both endpoints; for \( m \leq -1 \) the series diverges at both endpoints; and for \(-1 < m < 0\) the series converges conditionally at \( x = 1 \) and diverges at \( x = -1 \).
QUICK CHECK EXERCISES 9.9  (See page 677 for answers.)

1. \( \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \)
2. \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \)
3. \( \ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} \) for \( x \) in the interval ________.

3–10 Approximate the specified function value as indicated and check your work by comparing your answer to the function value produced directly by your calculating utility.

3. Approximate \( \sin 4^\circ \) to five decimal-place accuracy using both of the methods given in Example 2.
4. Approximate \( \cos 3^\circ \) to three decimal-place accuracy using both of the methods given in Example 2.
5. Approximate \( \cos 0.1 \) to five decimal-place accuracy using the Maclaurin series for \( \cos x \).
6. Approximate \( \tan^{-1} 0.1 \) to three decimal-place accuracy using the Maclaurin series for \( \tan^{-1} x \).
7. Approximate \( \sin 85^\circ \) to four decimal-place accuracy using an appropriate Taylor series.
8. Approximate \( \cos(-175^\circ) \) to four decimal-place accuracy using a Taylor series.
9. Approximate \( \sinh 0.5 \) to three decimal-place accuracy using the Maclaurin series for \( \sinh x \).
10. Approximate \( \cosh 0.1 \) to three decimal-place accuracy using the Maclaurin series for \( \cosh x \).
11. (a) Use Formula (12) in the text to find a series that converges to \( \ln 1.25 \).
   (b) Approximate \( \ln 1.25 \) using the first two terms of the series. Round your answer to three decimal places, and compare the result to that produced directly by your calculating utility.
12. (a) Use Formula (12) to find a series that converges to \( \ln 3 \).
   (b) Approximate \( \ln 3 \) using the first two terms of the series. Round your answer to three decimal places, and compare the result to that produced directly by your calculating utility.

FOCUS ON CONCEPTS

13. (a) Use the Maclaurin series for \( \tan^{-1} x \) to approximate \( \tan^{-1} \frac{1}{2} \) and \( \tan^{-1} \frac{1}{3} \) to three decimal-place accuracy.
   (b) Use the results in part (a) and Formula (16) to approximate \( \pi \).
   (c) Would you be willing to guarantee that your answer in part (b) is accurate to three decimal places? Explain your reasoning.
   (d) Compare your answer in part (b) to that produced by your calculating utility.

14. The purpose of this exercise is to show that the Taylor series of a function \( f \) may possibly converge to a value different from \( f(x) \) for certain values of \( x \). Let

\[
   f(x) = \begin{cases} 
   e^{-1/x^2}, & x \neq 0 \\
   0, & x = 0 
   \end{cases}
\]

(a) Use the definition of a derivative to show that \( f'(0) = 0 \).
   (b) With some difficulty it can be shown that if \( n \geq 2 \), then \( f^{(n)}(0) = 0 \). Accepting this fact, show that the Maclaurin series of \( f \) converges for all \( x \) but converges to \( f(x) \) only at \( x = 0 \).

15. (a) Find an upper bound on the error that can result if \( \cos x \) is approximated by \( 1 - (x^2/2!) + (x^4/4!) \) over the interval \([-0.2, 0.2]\).
   (b) Check your answer in part (a) by graphing 

\[
   |\cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)|
\]

over the interval.

16. (a) Find an upper bound on the error that can result if \( \ln(1 + x) \) is approximated by \( x \) over the interval \([-0.01, 0.01]\).
   (b) Check your answer in part (a) by graphing 

\[
   |\ln(1 + x) - x|
\]

over the interval.
17. Use Formula (17) for the binomial series to obtain the Maclaurin series for
(a) \( \frac{1}{1+x} \)  \hspace{1cm} (b) \( \sqrt{1+x} \) \hspace{1cm} (c) \( \frac{1}{(1+x)^3} \)

18. If \( m \) is any real number, and \( k \) is a nonnegative integer, then we define the binomial coefficient
\[
\binom{m}{k} = \frac{m(m-1)(m-2) \cdots (m-k+1)}{k!}
\]
for \( k \geq 1 \). Express Formula (17) in the text in terms of binomial coefficients.

19. In this exercise we will use the Remainder Estimation Theorem to determine the number of terms that are required in Formula (14) to approximate \( \ln 2 \) to five decimal-place accuracy. For this purpose let
\[ f(x) = \ln \left(1 + \frac{1}{x} \right) = \ln(1 + x) - \ln(1 - x) \quad (1 < x < 1) \]
(a) Show that \( f^{(n+1)}(x) = n! \left[ \frac{(-1)^n}{(1+x)^{n+1}} + \frac{1}{(1-x)^{n+1}} \right] \)
(b) Use the triangle inequality [Theorem 0.1.4(d)] to show that
\[ |f^{(n+1)}(x)| \leq n! \left[ \frac{1}{(1+x)^{n+1}} + \frac{1}{(1-x)^{n+1}} \right] \]
(c) Since we want to achieve five decimal-place accuracy, our goal is to choose \( n \) so that the absolute value of the \( n \)th remainder at \( x = \frac{1}{2} \) does not exceed the value 0.000005 = \( 0.5 \times 10^{-5} \); that is, \( |R_n\left(\frac{1}{2}\right)| \leq 0.000005 \). Use the Remainder Estimation Theorem to show that this condition will be satisfied if \( n \) is chosen so that
\[ M \left( \frac{1}{3} \right)^{n+1} \leq 0.000005 \]
where \( |f^{(n+1)}(x)| \leq M \) on the interval \([0, \frac{1}{2}]\).
(d) Use the result in part (b) to show that \( M \) can be taken as
\[ M = n! \left[ 1 + \frac{1}{\left(\frac{1}{2}\right)^{n+1}} \right] \]
(e) Use the results in parts (c) and (d) to show that five decimal-place accuracy will be achieved if \( n \) satisfies
\[ \frac{1}{n+1} \left[ \frac{1}{\left(\frac{1}{2}\right)^{n+1}} + \frac{1}{\left(\frac{1}{2}\right)^{n+1}} \right] \leq 0.000005 \]
and then that the smallest value of \( n \) that satisfies this condition is \( n = 13 \).

20. Use Formula (12) and the method of Exercise 19 to approximate \( \ln \left(\frac{3}{2}\right) \) to five decimal-place accuracy. Then check your work by comparing your answer to that produced directly by your calculating utility.

21. Prove: The Taylor series for \( \cos x \) about any value \( x = x_0 \) converges to \( \cos x \) for all \( x \).

22. Prove: The Taylor series for \( \sin x \) about any value \( x = x_0 \) converges to \( \sin x \) for all \( x \).

23. Research has shown that the proportion \( p \) of the population with IQs (intelligence quotients) between 100 and 110.

24. (a) In 1706 the British astronomer and mathematician John Machin discovered the following formula for \( \pi/4 \), called Machin’s formula:
\[ \pi = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} \]
Use a CAS to approximate \( \pi/4 \) using Machin’s formula to 25 decimal places.
(b) In 1914 the brilliant Indian mathematician Srinivasa Ramanujan (1887–1920) showed that
\[ \frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26,390k)}{(k!)^4 3964k} \]
Use a CAS to compute the first four partial sums in Ramanujan’s formula.

---

**Quick Check Answers 9.9**

1. \( (-1)^k \frac{x^{2k}}{(2k)!} \)
2. \( \frac{x^4}{k!} \)
3. \( (-1)^{k+1} \frac{x^k}{k} (-1, 1] \)
4. \( \frac{m(m-1) \cdots (m-k+1)}{k!} \frac{x^k}{k^1} \)
9.10 DIFFERENTIATING AND INTEGRATING POWER SERIES; MODELING WITH TAYLOR SERIES

In this section we will discuss methods for finding power series for derivatives and integrals of functions, and we will discuss some practical methods for finding Taylor series that can be used in situations where it is difficult or impossible to find the series directly.

### DIFFERENTIATING POWER SERIES

We begin by considering the following problem.

#### 9.10.1 Problem

Suppose that a function \( f \) is represented by a power series on an open interval. How can we use the power series to find the derivative of \( f \) on that interval?

The solution to this problem can be motivated by considering the Maclaurin series for \( \sin x \):

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (\infty < x < \infty)
\]

Of course, we already know that the derivative of \( \sin x \) is \( \cos x \); however, we are concerned here with using the Maclaurin series to deduce this. The solution is easy—all we need to do is differentiate the Maclaurin series term by term and observe that the resulting series is the Maclaurin series for \( \cos x \):

\[
\frac{d}{dx} \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right] = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots
\]

\[
= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x
\]

Here is another example.

\[
\frac{d}{dx} [e^x] = \frac{d}{dx} \left[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right]
\]

\[
= 1 + 2 \frac{x}{2!} + 3 \frac{x^2}{3!} + 4 \frac{x^3}{4!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x
\]

The preceding computations suggest that if a function \( f \) is represented by a power series on an open interval, then a power series representation of \( f' \) on that interval can be obtained by differentiating the power series for \( f \) term by term. This is stated more precisely in the following theorem, which we give without proof.

#### 9.10.2 Theorem (Differentiation of Power Series)

Suppose that a function \( f \) is represented by a power series in \( x - x_0 \) that has a nonzero radius of convergence \( R \); that is,

\[
f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \quad (x_0 - R < x < x_0 + R)
\]

Then:

(a) The function \( f \) is differentiable on the interval \( (x_0 - R, x_0 + R) \).

(b) If the power series representation for \( f \) is differentiated term by term, then the resulting series has radius of convergence \( R \) and converges to \( f' \) on the interval \( (x_0 - R, x_0 + R) \); that is,

\[
f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx} [c_k (x - x_0)^k] \quad (x_0 - R < x < x_0 + R)
\]
9.10 Differentiating and Integrating Power Series; Modeling with Taylor Series

This theorem has an important implication about the differentiability of functions that are represented by power series. According to the theorem, the power series for \( f' \) has the same radius of convergence as the power series for \( f \), and this means that the theorem can be applied to \( f' \) as well as \( f \). However, if we do this, then we conclude that \( f' \) is differentiable on the interval \((x_0 - R, x_0 + R)\), and the power series for \( f'' \) has the same radius of convergence as the power series for \( f \) and \( f' \). We can now repeat this process ad infinitum, applying the theorem successively to \( f'' \), \( f''' \), \ldots, \( f^{(n)} \), \ldots to conclude that \( f \) has derivatives of all orders on the interval \((x_0 - R, x_0 + R)\). Thus, we have established the following result.

9.10.3 Theorem

If a function \( f \) can be represented by a power series in \( x - x_0 \) with a nonzero radius of convergence \( R \), then \( f \) has derivatives of all orders on the interval \((x_0 - R, x_0 + R)\).

In short, it is only the most “well-behaved” functions that can be represented by power series; that is, if a function \( f \) does not possess derivatives of all orders on an interval \((x_0 - R, x_0 + R)\), then it cannot be represented by a power series in \( x - x_0 \) on that interval.

Example 1

In Section 9.8, we showed that the Bessel function \( J_0(x) \), represented by the power series

\[
J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)^2}
\]

has radius of convergence \(+\infty\) [see Formula (4) of that section and the related discussion]. Thus, \( J_0(x) \) has derivatives of all orders on the interval \((-\infty, +\infty)\), and these can be obtained by differentiating the series term by term. For example, if we write (1) as

\[
J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)^2}
\]

and differentiate term by term, we obtain

\[
J_0'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{2^{2k} (k!)^2} = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{2^{2k-1} k! (k-1)!}
\]

REMARK

The computations in this example use some techniques that are worth noting. First, when a power series is expressed in sigma notation, the formula for the general term of the series will often not be of a form that can be used for differentiating the constant term. Thus, if the series has a nonzero constant term, as here, it is usually a good idea to split it off from the summation before differentiating. Second, observe how we simplified the final formula by canceling the factor \( k \) from one of the factorials in the denominator. This is a standard simplification technique.

INTEGRATING POWER SERIES

Since the derivative of a function that is represented by a power series can be obtained by differentiating the series term by term, it should not be surprising that an antiderivative of a function represented by a power series can be obtained by integrating the series term by term. For example, we know that \( \sin x \) is an antiderivative of \( \cos x \). Here is how this result
can be obtained by integrating the Maclaurin series for \( \cos x \) term by term:

\[
\int \cos x \, dx = \int \left[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right] \, dx
\]

\[
= \left[ x - \frac{x^3}{3(2!)} + \frac{x^5}{5(4!)} - \frac{x^7}{7(6!)} + \cdots \right] + C
\]

\[
= \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right] + C = \sin x + C
\]

The same idea applies to definite integrals. For example, by direct integration we have

\[
\int_0^1 \frac{dx}{1 + x^2} = \tan^{-1} x \bigg|_0^1 = \tan^{-1} 1 - \tan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}
\]

and we will show later in this section that

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]

Thus,

\[
\int_0^1 \frac{dx}{1 + x^2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]

Here is how this result can be obtained by integrating the Maclaurin series for \( 1/(1 + x^2) \) term by term (see Table 9.9.1):

\[
\int_0^1 \frac{dx}{1 + x^2} = \int_0^1 [1 - x^2 + x^4 - x^6 + \cdots] \, dx
\]

\[
= \left[ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right] \bigg|_0^1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]

The preceding computations are justified by the following theorem, which we give without proof.

**THEOREM (Integration of Power Series)** Suppose that a function \( f \) is represented by a power series in \( x - x_0 \) that has a nonzero radius of convergence \( R \); that is,

\[
f(x) = \sum_{k=0}^\infty c_k (x - x_0)^k \quad (x_0 - R < x < x_0 + R)
\]

(a) If the power series representation of \( f \) is integrated term by term, then the resulting series has radius of convergence \( R \) and converges to an antiderivative for \( f(x) \) on the interval \( (x_0 - R, x_0 + R) \); that is,

\[
\int f(x) \, dx = \sum_{k=0}^\infty \left[ \frac{c_k}{k+1} (x - x_0)^{k+1} \right] + C \quad (x_0 - R < x < x_0 + R)
\]

(b) If \( \alpha \) and \( \beta \) are points in the interval \( (x_0 - R, x_0 + R) \), and if the power series representation of \( f \) is integrated term by term from \( \alpha \) to \( \beta \), then the resulting series converges absolutely on the interval \( (x_0 - R, x_0 + R) \) and

\[
\int_\alpha^\beta f(x) \, dx = \sum_{k=0}^\infty \left[ \int_\alpha^\beta c_k (x - x_0)^k \, dx \right]
\]
9.10 Differentiating and Integrating Power Series; Modeling with Taylor Series

**POWER SERIES REPRESENTATIONS MUST BE TAYLOR SERIES**

For many functions it is difficult or impossible to find the derivatives that are required to obtain a Taylor series. For example, to find the Maclaurin series for \( \frac{1}{1 + x^2} \) directly would require some tedious derivative computations (try it). A more practical approach is to substitute \(-x^2\) for \(x\) in the geometric series

\[ \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1) \]

to obtain

\[ \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \]

However, there are two questions of concern with this procedure:

- Where does the power series that we obtained for \( \frac{1}{1 + x^2} \) actually converge to \( \frac{1}{1 + x^2} \)?
- How do we know that the power series we have obtained is actually the Maclaurin series for \( \frac{1}{1 + x^2} \)?

The first question is easy to resolve. Since the geometric series converges to \( \frac{1}{1 - x} \) if \(|x| < 1\), the second series will converge to \( \frac{1}{1 + x^2} \) if \(|-x^2| < 1\) or \(|x^2| < 1\). However, this is true if and only if \(|x| < 1\), so the power series we obtained for the function \( \frac{1}{1 + x^2} \) converges to this function if \(-1 < x < 1\).

The second question is more difficult to answer and leads us to the following general problem.

**9.10.5 Problem** Suppose that a function \( f \) is represented by a power series in \( x - x_0 \) that has a nonzero radius of convergence. What relationship exists between the given power series and the Taylor series for \( f \) about \( x = x_0 \)?

The answer is that they are the same; and here is the theorem that proves it.

**9.10.6 Theorem** If a function \( f \) is represented by a power series in \( x - x_0 \) on some open interval containing \( x_0 \), then that power series is the Taylor series for \( f \) about \( x = x_0 \).

**Proof** Suppose that

\[ f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_k(x - x_0)^k + \cdots \]

for all \( x \) in some open interval containing \( x_0 \). To prove that this is the Taylor series for \( f \) about \( x = x_0 \), we must show that

\[ c_k = \frac{f^{(k)}(x_0)}{k!} \quad \text{for} \quad k = 0, 1, 2, 3, \ldots \]

However, the assumption that the series converges to \( f(x) \) on an open interval containing \( x_0 \) ensures that it has a nonzero radius of convergence \( R \); hence we can differentiate term
by term in accordance with Theorem 9.10.2. Thus,

\[ f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + c_4(x - x_0)^4 + \cdots \]

\[ f'(x) = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + 4c_4(x - x_0)^3 + \cdots \]

\[ f''(x) = 2!c_2 + (3 \cdot 2)c_3(x - x_0) + (4 \cdot 3)c_4(x - x_0)^2 + \cdots \]

\[ f'''(x) = 3!c_3 + (4 \cdot 3 \cdot 2)c_4(x - x_0) + \cdots \]

On substituting \( x = x_0 \), all the powers of \( x - x_0 \) drop out, leaving

\[ f(x_0) = c_0, \quad f'(x_0) = c_1, \quad f''(x_0) = 2!c_2, \quad f'''(x_0) = 3!c_3, \ldots \]

from which we obtain

\[ c_0 = f(x_0), \quad c_1 = f'(x_0), \quad c_2 = \frac{f''(x_0)}{2!}, \quad c_3 = \frac{f'''(x_0)}{3!}, \ldots \]

which shows that the coefficients \( c_0, c_1, c_2, c_3, \ldots \) are precisely the coefficients in the Taylor series about \( x_0 \) for \( f(x) \).

### SOME PRACTICAL WAYS TO FIND TAYLOR SERIES

**Example 2**  
Find Taylor series for the given functions about the given \( x_0 \).

(a) \( e^{-x^2} \), \( x_0 = 0 \)  
(b) \( \ln x \), \( x_0 = 1 \)  
(c) \( \frac{1}{x} \), \( x_0 = 1 \)

**Solution (a).** The simplest way to find the Maclaurin series for \( e^{-x^2} \) is to substitute \(-x^2\) for \( x \) in the Maclaurin series

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \]  

(3)

to obtain

\[ e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots \]

Since (3) converges for all values of \( x \), so will the series for \( e^{-x^2} \).

**Solution (b).** We begin with the Maclaurin series for \( \ln(1 + x) \), which can be found in Table 9.9.1:

\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{\((-1 < x \leq 1\))} \]

Substituting \( x - 1 \) for \( x \) in this series gives

\[ \ln(1 + [x - 1]) = \ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots \]  

(4)

Since the original series converges when \(-1 < x \leq 1\), the interval of convergence for (4) will be \(-1 < x - 1 \leq 1\) or, equivalently, \(0 < x \leq 2\).

**Solution (c).** Since \( 1/x \) is the derivative of \( \ln x \), we can differentiate the series for \( \ln x \) found in (b) to obtain

\[ \frac{1}{x} = 1 - \frac{2(x - 1)}{2} + \frac{3(x - 1)^2}{3} - \frac{4(x - 1)^3}{4} + \cdots \]

\[ = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots \]  

(5)
9.10 Differentiating and Integrating Power Series; Modeling with Taylor Series

By Theorem 9.10.2, we know that the radius of convergence for (5) is the same as that for (4), which is $R = 1$. Thus the interval of convergence for (5) must be at least $0 < x < 2$. Since the behaviors of (4) and (5) may differ at the endpoints $x = 0$ and $x = 2$, those must be checked separately. When $x = 0$, (5) becomes

$$1 = (-1) + (-1)^2 - (-1)^3 + \cdots = 1 + 1 + 1 + \cdots$$

which diverges by the divergence test. Similarly, when $x = 2$, (5) becomes

$$1 - 1 + 1^2 - 1^3 + \cdots = 1 - 1 - 1 + \cdots$$

which also diverges by the divergence test. Thus the interval of convergence for (5) is $0 < x < 2$.

Example 3 Find the Maclaurin series for $\tan^{-1}x$.

Solution. It would be tedious to find the Maclaurin series directly. A better approach is to start with the formula

$$\int \frac{1}{1 + x^2} \, dx = \tan^{-1}x + C$$

and integrate the Maclaurin series

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \quad (-1 < x < 1)$$

term by term. This yields

$$\tan^{-1}x + C = \int \frac{1}{1 + x^2} \, dx = \int [1 - x^2 + x^4 - x^6 + x^8 - \cdots] \, dx$$

or

$$\tan^{-1}x = \left[ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \right] - C$$

The constant of integration can be evaluated by substituting $x = 0$ and using the condition $\tan^{-1}0 = 0$. This gives $C = 0$, so that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad (-1 < x < 1) \quad (6)$$

Remark Observe that neither Theorem 9.10.2 nor Theorem 9.10.3 addresses what happens at the endpoints of the interval of convergence. However, it can be proved that if the Taylor series for $f$ about $x = x_0$ converges to $f(x)$ for all $x$ in the interval $(x_0 - R, x_0 + R)$, and if the Taylor series converges at the right endpoint $x_0 + R$, then the value that it converges to at that point is the limit of $f(x)$ as $x \to x_0 + R$ from the right; and if the Taylor series converges at the left endpoint $x_0 - R$, then the value that it converges to at that point is the limit of $f(x)$ as $x \to x_0 - R$ from the right.

For example, the Maclaurin series for $\tan^{-1}x$ given in (6) converges at both $x = -1$ and $x = 1$, since the hypotheses of the alternating series test (Theorem 9.6.1) are satisfied at those points. Thus, the continuity of $\tan^{-1}x$ on the interval $[-1, 1]$ implies that at $x = 1$ the Maclaurin series converges to

$$\lim_{x \to 1^+} \tan^{-1}x = \tan^{-1}1 = \frac{\pi}{4}$$

and at $x = -1$ it converges to

$$\lim_{x \to -1^+} \tan^{-1}x = \tan^{-1}(-1) = -\frac{\pi}{4}$$

This shows that the Maclaurin series for $\tan^{-1}x$ actually converges to $\tan^{-1}x$ on the closed interval $-1 \leq x \leq 1$. Moreover, the convergence at $x = 1$ establishes Formula (2).
APPROXIMATING DEFINITE INTEGRALS USING TAYLOR SERIES

Taylor series provide an alternative to Simpson’s rule and other numerical methods for approximating definite integrals.

Example 4

Approximate the integral

\[ \int_0^1 e^{-x^2} \, dx \]

to three decimal-place accuracy by expanding the integrand in a Maclaurin series and integrating term by term.

Solution.

We found in Example 2(a) that the Maclaurin series for \( e^{-x^2} \) is

\[ e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots \]

Therefore,

\[
\int_0^1 e^{-x^2} \, dx = \int_0^1 \left[ 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots \right] \, dx
\]

\[
= \left[ x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} - \cdots \right]_0^1
\]

\[
= 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \cdots
\]

\[
= \sum_{k=0}^\infty \frac{(-1)^k}{(2k + 1)k!}
\]

Since this series clearly satisfies the hypotheses of the alternating series test (Theorem 9.6.1), it follows from Theorem 9.6.2 that if we approximate the integral by \( s_n \) (the \( n \)th partial sum of the series), then

\[
\left| \int_0^1 e^{-x^2} \, dx - s_n \right| \leq \frac{1}{(2(n + 1) + 1)(n + 1)!} = \frac{1}{(2n + 3)(n + 1)!}
\]

Thus, for three decimal-place accuracy we must choose \( n \) such that

\[
\frac{1}{(2n + 3)(n + 1)!} \leq 0.0005 = 5 \times 10^{-4}
\]

With the help of a calculating utility you can show that the smallest value of \( n \) that satisfies this condition is \( n = 5 \). Thus, the value of the integral to three decimal-place accuracy is

\[
\int_0^1 e^{-x^2} \, dx \approx 1 - \frac{1}{5} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} - \frac{1}{11 \cdot 5!} \approx 0.747
\]

As a check, a calculator with a built-in numerical integration capability produced the approximation 0.746824, which agrees with our result when rounded to three decimal places.

FINDING TAYLOR SERIES BY MULTIPLICATION AND DIVISION

The following examples illustrate some algebraic techniques that are sometimes useful for finding Taylor series.
9.10 Differentiating and Integrating Power Series; Modeling with Taylor Series

Example 5  Find the first three nonzero terms in the Maclaurin series for the function \( f(x) = e^{-x^2} \tan^{-1} x \).

Solution. Using the series for \( e^{-x^2} \) and \( \tan^{-1} x \) obtained in Examples 2 and 3 gives

\[
e^{-x^2} \tan^{-1} x = \left( 1 - \frac{x^2}{2} + \frac{x^4}{2} - \cdots \right) \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \right)
\]

Multiplying, as shown in the margin, we obtain

\[
e^{-x^2} \tan^{-1} x = x - \frac{4x^3}{3} + \frac{31x^5}{30} - \cdots
\]

More terms in the series can be obtained by including more terms in the factors. Moreover, one can prove that a series obtained by this method converges at each point in the intersection of the intervals of convergence of the factors (and possibly on a larger interval). Thus, we can be certain that the series we have obtained converges for all \( x \) in the interval \( -1 \leq x \leq 1 \) (why?).

Example 6  Find the first three nonzero terms in the Maclaurin series for \( \tan x \).

Solution. Using the first three terms in the Maclaurin series for \( \sin x \) and \( \cos x \), we can express \( \tan x \) as

\[
\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}
\]

Dividing, as shown in the margin, we obtain

\[
\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots
\]

Modeling Physical Laws with Taylor Series

Taylor series provide an important way of modeling physical laws. To illustrate the idea we will consider the problem of modeling the period of a simple pendulum (Figure 9.10.1). As explained in Chapter 7 Making Connections Exercise 5, the period \( T \) of such a pendulum is given by

\[
T = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi
\]

where

\[
L = \text{length of the supporting rod}
\]

\[
g = \text{acceleration due to gravity}
\]

\[
k = \sin(\theta_0/2), \text{ where } \theta_0 \text{ is the initial angle of displacement from the vertical}
\]

The integral, which is called a complete elliptic integral of the first kind, cannot be expressed in terms of elementary functions and is often approximated by numerical methods. Unfortunately, numerical values are so specific that they often give little insight into general physical principles. However, if we expand the integrand of (7) in a series and integrate term by term, then we can generate an infinite series that can be used to construct various mathematical models for the period \( T \) that give a deeper understanding of the behavior of the pendulum.
EXERCISE SET 9.10

1. In each part, obtain the Maclaurin series for the function by making an appropriate substitution in the Maclaurin series for 1/(1 − x). Include the general term in your answer, and state the radius of convergence of the series.
   (a) \( \frac{1}{1 + x} \)  
   (b) \( \frac{1}{1 - x^2} \)  
   (c) \( \frac{1}{1 - 2x} \)  
   (d) \( \frac{1}{2 - x} \)

2. In each part, obtain the Maclaurin series for the function by making an appropriate substitution in the Maclaurin series for \( \ln(1 + x) \). Include the general term in your answer, and state the radius of convergence of the series.
   (a) \( \ln(1 - x) \)  
   (b) \( \ln(1 + x^2) \)  
   (c) \( \ln(1 + 2x) \)  
   (d) \( \ln(2 + x) \)

3. In each part, obtain the first four nonzero terms of the Maclaurin series for the function by making an appropriate substitution in one of the binomial series obtained in Example 4 of Section 9.9.
   (a) \( (2 + x)^{-1/2} \)  
   (b) \( (1 - x^2)^{-2} \)
9.10 Differentiating and Integrating Power Series; Modeling with Taylor Series

4. (a) Use the Maclaurin series for \(1/(1-x)\) to find the Maclaurin series for \(1/(a-x)\), where \(a \neq 0\), and state the radius of convergence of the series.

(b) Use the binomial series for \(1/(1+x)^2\) obtained in Example 4 of Section 9.9 to find the first four nonzero terms in the Maclaurin series for \(1/(a+x)^2\), where \(a \neq 0\), and state the radius of convergence of the series.

5–8 Find the first four nonzero terms of the Maclaurin series for the function by making an appropriate substitution in a known Maclaurin series and performing any algebraic operations that are required. State the radius of convergence of the series.

5. (a) \(\sin 2x\) (b) \(e^{-2x}\) (c) \(e^x\) (d) \(x^2 \cos \pi x\)

6. (a) \(\cos 2x\) (b) \(x^2 e^x\) (c) \(xe^{-x}\) (d) \(\sin(x^2)\)

7. (a) \(x^2 / (1 + 3x)\) (b) \(x \sinh 2x\) (c) \(x(1-x^2)^{3/2}\)

8. (a) \(x / x^2 - 1\) (b) \(3 \cosh(x^2)\) (c) \(x / (1 + 2x)^x\)

9–10 Find the first four nonzero terms of the Maclaurin series for the function by using an appropriate trigonometric identity or property of logarithms and then substituting in a known Maclaurin series.

9. (a) \(\sin^2 x\) (b) \(\ln[(1 + x^3)²]\)

10. (a) \(\cos^2 x\) (b) \(\ln(1 + x)\)

11. (a) Use a known Maclaurin series to find the Taylor series of \(1/x\) about \(x = 1\) by expressing this function as

\[
\frac{1}{x} = 1 - (1 - x)
\]

(b) Find the interval of convergence of the Taylor series.

12. Use the method of Exercise 11 to find the Taylor series of \(1/x\) about \(x = x_0\), and state the interval of convergence of the Taylor series.

13–14 Find the first four nonzero terms of the Maclaurin series for the function by multiplying the Maclaurin series of the factors.

13. (a) \(e^x \sin x\) (b) \(\sqrt{1+x} \ln(1+x)\)

14. (a) \(e^{-x^2} \cos x\) (b) \((1+x)^{3/4} / (1+x)^{1/3}\)

15–16 Find the first four nonzero terms of the Maclaurin series for the function by dividing appropriate Maclaurin series.

15. (a) \(\sec x\) (b) \(\frac{\sin x}{e^x}\)

16. (a) \(\tan^{-1} x / 1 + x\) (b) \(\frac{\ln(1+x)}{1-x}\)

17. Use the Maclaurin series for \(e^x\) and \(e^{-x}\) to derive the Maclaurin series for \(\sin x\) and \(\cosh x\). Include the general terms in your answers and state the radius of convergence of each series.

18. Use the Maclaurin series for \(\sin x\) and \(\cosh x\) to obtain the first four nonzero terms in the Maclaurin series for \(\tanh x\).

19–20 Find the first five nonzero terms of the Maclaurin series for the function by using partial fractions and a known Maclaurin series.

19. \[
\frac{4x-2}{x^2-1}
\]

20. \[
\frac{x^3 + x^2 + 2x - 2}{x^2 - 1}
\]

21–22 Confirm the derivative formula by differentiating the appropriate Maclaurin series term by term.

21. (a) \(\frac{d}{dx} \cos x = -\sin x\) (b) \(\frac{d}{dx} [\ln(1+x)] = \frac{1}{1+x}\)

22. (a) \(\frac{d}{dx} \sinh x = \cosh x\) (b) \(\frac{d}{dx} [\tan^{-1} x] = \frac{1}{1+x^2}\)

23–24 Confirm the integration formula by integrating the appropriate Maclaurin series term by term.

23. (a) \(\int e^x \, dx = e^x + C\)

(b) \(\int \sinh x \, dx = \cosh x + C\)

24. (a) \(\int \sin x \, dx = -\cos x + C\)

(b) \(\int \frac{1}{1+x} \, dx = \ln(1+x) + C\)

25. Consider the series

\[
\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)(k+2)}
\]

Determine the intervals of convergence for this series and for the series obtained by differentiating this series term by term.

26. Consider the series

\[
\sum_{k=1}^{\infty} \frac{(-3)^k}{k} x^k
\]

Determine the intervals of convergence for this series and for the series obtained by integrating this series term by term.

27. (a) Use the Maclaurin series for \(1/(1-x)\) to find the Maclaurin series for

\[
f(x) = \frac{x}{1-x^2}
\]

(b) Use the Maclaurin series obtained in part (a) to find \(f^{(5)}(0)\) and \(f^{(6)}(0)\).

(c) What can you say about the value of \(f^{(n)}(0)\)?

28. Let \(f(x) = x^2 \cos 2x\). Use the method of Exercise 27 to find \(f^{(99)}(0)\).

29–30 The limit of an indeterminate form as \(x \to x_0\) can sometimes be found by expanding the functions involved in Taylor series about \(x = x_0\) and taking the limit of the series term by term. Use this method to find the limits in these exercises.

29. (a) \(\lim_{x \to 0} \frac{\sin x}{x}\) (b) \(\lim_{x \to 0} \frac{\tan^{-1} x - x}{x^3}\)

30. (a) \(\lim_{x \to 0} \frac{1 - \cos x}{\sin x}\) (b) \(\lim_{x \to 0} \frac{\ln(1+x) - \sin 2x}{x}\)
31–34 Use Maclaurin series to approximate the integral to three decimal-place accuracy.

31. \[ \int_0^1 \sin(x^2) \, dx \]

32. \[ \int_0^{1/2} \tan^{-1}(2x^2) \, dx \]

33. \[ \int_0^{0.2} \sqrt{1 + x^4} \, dx \]

34. \[ \int_0^{1/2} \frac{dx}{\sqrt{x^2 + 1}} \]

35. (a) Find the Maclaurin series for \( e^{ix} \). What is the radius of convergence?

(b) Explain two different ways to use the Maclaurin series for \( e^{ix} \) to find a series for \( x^3 e^{ix} \). Confirm that both methods produce the same series.

36. (a) Differentiate the Maclaurin series for \( 1/(1 - x) \), and use the result to show that

\[
\sum_{k=1}^{\infty} k x^k = \frac{x}{(1 - x)^2} \quad \text{for} \ 1 < x < 1
\]

(b) Integrate the Maclaurin series for \( 1/(1 - x) \), and use the result to show that

\[
\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1 - x) \quad \text{for} \ 1 < x < 1
\]

(c) Use the result in part (b) to show that

\[
\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k} = \ln(1 + x) \quad \text{for} \ -1 < x < 1
\]

(d) Show that the series in part (c) converges if \( x = 1 \).

(e) Use the remark following Example 3 to show that

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = \ln(1 + x) \quad \text{for} \ -1 < x \leq 1
\]

37. Use the results in Exercise 36 to find the sum of the series.

(a) \[ \sum_{k=1}^{\infty} \frac{k}{3^k} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots \]

(b) \[ \sum_{k=1}^{\infty} \frac{1}{k(4^k)} = \frac{1}{4} + \frac{1}{2(4^2)} + \frac{1}{3(4^3)} + \frac{1}{4(4^4)} + \cdots \]

38. Use the results in Exercise 36 to find the sum of each series.

(a) \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \]

(b) \[ \sum_{k=1}^{\infty} \frac{(e - 1)^k}{ke^k} = \frac{e - 1}{e} + \frac{(e - 1)^2}{2(e^2)} - \frac{(e - 1)^3}{3(e^3)} + \cdots \]

39. (a) Use the relationship

\[ \int \frac{1}{\sqrt{4 + x^2}} \, dx = \sinh^{-1} x + C \]

(b) Express the series in sigma notation.

(c) What is the radius of convergence?

40. (a) Use the relationship

\[ \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C \]

(b) Express the series in sigma notation.

(c) What is the radius of convergence?

41. We showed by Formula (19) of Section 8.2 that if there are \( y_0 \) units of radioactive carbon-14 present at time \( t = 0 \), then the number of units present \( t \) years later is

\[ y(t) = y_0 e^{-0.00121t} \]

(a) Express \( y(t) \) as a Maclaurin series.

(b) Use the first two terms in the series to show that the number of units present after 1 year is approximately \( 0.999879 y_0 \).

(c) Compare this to the value produced by the formula for \( y(t) \).

42. Suppose that a simple pendulum with a length of \( L = 1 \) meter is given an initial displacement of \( \theta_0 = 5^\circ \) from the vertical.

(a) Approximate the period \( T \) of the pendulum using Formula (9) for the first-order model of \( T \). [Note: Take \( g = 9.8 \text{ m/s}^2 \).]

(b) Approximate the period of the pendulum using Formula (10) for the second-order model.

(c) Use the numerical integration capability of a CAS to approximate the period of the pendulum from Formula (7), and compare it to the values obtained in parts (a) and (b).
percentage does a person’s weight change in going from mean sea level to the top of Mt. Everest (29,028 ft)?

45. (a) Show that the Bessel function \( J_0(x) \) given by Formula (4) of Section 9.8 satisfies the differential equation \( x^2 y'' + xy' + (x^2 - 1)y = 0 \). (This is called the Bessel equation of order zero.)

(b) Show that the Bessel function \( J_1(x) \) given by Formula (5) of Section 9.8 satisfies the differential equation \( x^2 y'' + xy' + (x^2 - 1)y = 0 \). (This is called the Bessel equation of order one.)

(c) Show that \( J_0(x) = -J_1(x) \).

46. Prove: If the power series \( \sum_{k=0}^{\infty} a_k x^k \) and \( \sum_{k=0}^{\infty} b_k x^k \) have the same sum on an interval \((-r, r)\), then \( a_k = b_k \) for all values of \( k \).

47. Writing Evaluate the limit \( \lim_{x \to 0} \frac{x - \sin x}{x^3} \) in two ways: using L'Hôpital’s rule and by replacing \( \sin x \) by its Maclaurin series. Discuss how the use of a series can give qualitative information about how the value of an indeterminate limit is approached.

**Chapter 9 Review Exercises**

1. What is the difference between an infinite sequence and an infinite series?

2. What is meant by the sum of an infinite series?

3. (a) What is a geometric series? Give some examples of convergent and divergent geometric series.

   (b) What is a \( p \)-series? Give some examples of convergent and divergent \( p \)-series.

4. State conditions under which an alternating series is guaranteed to converge.

5. (a) What does it mean to say that an infinite series converges absolutely?

   (b) What relationship exists between convergence and absolute convergence of an infinite series?

6. State the Remainder Estimation Theorem, and describe some of its uses.

7. If a power series in \( x - x_0 \) has radius of convergence \( R \), what can you say about the set of \( x \)-values at which the series converges?

8. (a) Write down the formula for the Maclaurin series for \( f \) in sigma notation.

   (b) Write down the formula for the Taylor series for \( f \) about \( x = x_0 \) in sigma notation.

9. Are the following statements true or false? If true, state a theorem to justify your conclusion; if false, then give a counterexample.

   (a) If \( \sum u_k \) converges, then \( u_k \to 0 \) as \( k \to +\infty \).

   (b) If \( u_k \to 0 \) as \( k \to +\infty \), then \( \sum u_k \) converges.

   (c) If \( f(n) = a_n \) for \( n = 1, 2, 3, \ldots \) and if \( a_n \to L \) as \( n \to +\infty \), then \( f(x) \to L \) as \( x \to +\infty \).

   (d) If \( f(n) = a_n \) for \( n = 1, 2, 3, \ldots \) and if \( f(x) \to L \) as \( x \to +\infty \), then \( a_n \to L \) as \( n \to +\infty \).

   (e) If \( 0 < a_n < 1 \), then \( [a_n] \) converges.

   (f) If \( 0 < u_k < 1 \), then \( \sum u_k \) converges.

   (g) If \( \sum u_k \) and \( \sum v_k \) converge, then \( \sum (u_k + v_k) \) diverges.

   (h) If \( \sum u_k \) and \( \sum v_k \) diverge, then \( \sum (u_k - v_k) \) converges.

   (i) If \( 0 \leq u_k \leq v_k \) and \( \sum v_k \) converges, then \( \sum u_k \) converges.

   (j) If \( 0 \leq u_k \leq v_k \) and \( \sum u_k \) diverges, then \( \sum v_k \) diverges.

   (k) If an infinite series converges, then it converges absolutely.

   (l) If an infinite series diverges absolutely, then it diverges.

10. State whether each of the following is true or false. Justify your answers.

   (a) The function \( f(x) = x^{1/3} \) has a Maclaurin series.

   (b) \( 1 + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{8} + \cdots = 1 \)

   (c) \( 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \cdots = 1 \)

11. Find the general term of the sequence, starting with \( n = 1 \), determine whether the sequence converges, and if so find its limit.

   (a) \( \frac{3}{1^2 - 1^2} + \frac{3}{2^2 - 2^2} + \frac{3}{4^2 - 3^2} + \cdots \)

   (b) \( \frac{1}{3} + \frac{1}{7} + \frac{1}{9} + \cdots \)

12. Suppose that the sequence \( \{a_k\} \) is defined recursively by

   \[ a_0 = c, \quad a_{k+1} = \sqrt{a_k} \]

   Assuming that the sequence converges, find its limit if

   (a) \( c = \frac{1}{2} \)  \quad (b) \( c = \frac{1}{2} \)

13. Show that the sequence is eventually strictly monotonic.

   (a) \( \left\{(n - 10)^{k^2}\right\}_{k=0}^{\infty} \)

   (b) \( \left\{\frac{100}{(2n)!/(n!)^2}\right\}_{n=1}^{\infty} \)

14. (a) Give an example of a bounded sequence that diverges.

   (b) Give an example of a monotonic sequence that diverges.

15–20 Use any method to determine whether the series converge.
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15. (a) \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) (b) \( \sum_{k=1}^{\infty} \frac{1}{5k + 1} \)

16. (a) \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 + k} \) (b) \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (k + 2)}{3k - 1} \)

17. (a) \( \sum_{k=1}^{\infty} \frac{1}{k^3 + 2k + 1} \) (b) \( \sum_{k=1}^{\infty} \frac{1}{(3 + k)^{2/5}} \)

18. (a) \( \sum_{k=1}^{\infty} k \ln k \) (b) \( \sum_{k=1}^{\infty} 8k^2 + 5k + 1 \)

19. (a) \( \sum_{k=1}^{\infty} \frac{9}{k^{1/2}} \) (b) \( \sum_{k=1}^{\infty} \frac{\cos(1/k)}{k^2} \)

20. (a) \( \sum_{k=1}^{\infty} \frac{k^{-1/2}}{2 + \sin^2 k} \) (b) \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2 + 1} \)

21. Find the exact error that results when the sum of the geometric series \( \sum_{k=0}^{\infty} (1/5)^k \) is approximated by the sum of the first 100 terms in the series.

22. Suppose that \( \sum_{k=1}^{\infty} u_k = 2 - \frac{1}{n} \). Find \( (a) u_{100} \) (b) \( \lim_{k \to \infty} u_k \) (c) \( \sum_{k=1}^{\infty} u_k \).

23. In each part, determine whether the series converges; if so, find its sum.
(a) \( \sum_{k=1}^{\infty} \frac{3 k^2 - 2}{k^2} \) (b) \( \sum_{k=1}^{\infty} \ln(k + 1) - \ln k \)
(c) \( \sum_{k=1}^{\infty} \frac{1}{k(k + 2)} \) (d) \( \sum_{k=1}^{\infty} [\tan^{-1}(k + 1) - \tan^{-1} k] \)

24. It can be proved that
\[ \lim_{n \to \infty} \frac{\sqrt{n!}}{n^{2/3}} = +\infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\sqrt{n!}}{n^{1/2}} = e \]
In each case, use these limits and the root test to determine whether the series converges.
(a) \( \sum_{k=0}^{\infty} \frac{2^k}{k!} \) (b) \( \sum_{k=0}^{\infty} \frac{k^k}{k!} \)

25. Let \( a, b, \) and \( p \) be positive constants. For which values of \( p \) does the series \( \sum_{k=1}^{\infty} \frac{1}{(a + bk)^p} \) converge?

26. Find the interval of convergence of \( \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{b^k} \) (\( b > 0 \))

27. (a) Show that \( k^b \geq k! \).
(b) Use the comparison test to show that \( \sum_{k=1}^{\infty} k^{-b} \) converges.
(c) Use the root test to show that the series converges.

28. Does the series \( 1 - \frac{1}{2^b} + \frac{3}{3^b} - \frac{4}{4^b} + \frac{5}{5^b} + \cdots \) converge? Justify your answer.

29. (a) Find the first five Maclaurin polynomials of the function \( p(x) = 1 - 7x + 5x^2 + 4x^3 \).
(b) Make a general statement about the Maclaurin polynomials of a polynomial of degree \( n \).

30. Show that the approximation
\[ \sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \]
is accurate to four decimal places if \( 0 \leq x \leq \pi/4 \).

31. Use a Maclaurin series and properties of alternating series to show that \( |\ln(1 + x) - x| \leq x^2/2 \) if \( 0 < x < 1 \).

32. Use Maclaurin series to approximate the integral
\[ \int_0^1 \frac{1 - \cos x}{x} \, dx \]
to three decimal-place accuracy.

33. In parts (a)–(d), find the sum of the series by associating it with some Maclaurin series.
(a) \( 2 + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \cdots \)
(b) \( \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots \)
(c) \( 1 - \frac{e^2}{2!} + \frac{e^4}{4!} - \frac{e^6}{6!} + \cdots \)
(d) \( 1 - \ln 3 + \frac{(\ln 3)^2}{2!} - \frac{(\ln 3)^3}{3!} + \cdots \)

34. In each part, write out the first four terms of the series, and then find the radius of convergence.
(a) \( \sum_{k=1}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots k}{1 \cdot 4 \cdot 7 \cdots (3k - 2)} x^k \)
(b) \( \sum_{k=1}^{\infty} \frac{(-1)^k}{1 \cdot 3 \cdot 5 \cdots (2k - 1)} x^{2k+1} \)

35. Use an appropriate Taylor series for \( \sqrt{3} \) to approximate \( \sqrt{28} \) to three decimal-place accuracy, and check your answer by comparing it to that produced directly by your calculating utility.

36. Differentiate the Maclaurin series for \( xe^x \) and use the result to show that
\( \sum_{k=0}^{\infty} \frac{k + 1}{k!} = 2e \)

37. Use the supplied Maclaurin series for \( \sin x \) and \( \cos x \) to find the first four nonzero terms of the Maclaurin series for the given functions.
\[ \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \]
\[ \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \]
(a) \( \sin x \cos x \) (b) \( \frac{1}{2} \sin 2x \)
CHAPTER 9 MAKING CONNECTIONS

1. As shown in the accompanying figure, suppose that lines $L_1$ and $L_2$ form an angle $\theta$, $0 < \theta < \pi/2$, at their point of intersection $P$. A point $P_0$ is chosen that is on $L_1$ and $a$ units from $P$. Starting from $P_0$ a zig-zag path is constructed by successively going back and forth between $L_1$ and $L_2$ along a perpendicular from one line to the other. Find the following sums in terms of $\theta$ and $a$.
   (a) $P_0P_1 + P_1P_2 + P_2P_3 + \cdots$
   (b) $P_0P_1 + P_1P_3 + P_3P_5 + \cdots$
   (c) $P_0P_1 + P_3P_4 + P_5P_6 + \cdots$

   ![Figure Ex-1](image)

2. (a) Find $A$ and $B$ such that
   \[ \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} = \frac{2^k A}{3^k - 2^k} + \frac{2^k B}{3^{k+1} - 2^{k+1}} \]
   (b) Use the result in part (a) to find a closed form for the $n$th partial sum of the series
   \[ \sum_{k=1}^{n} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} \]
   and then find the sum of the series.

   Source: This exercise is adapted from a problem that appeared in the Forty-Fifth Annual William Lowell Putnam Competition.

3. Show that the alternating $p$-series
   \[ 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots + \frac{1}{k^p} + \cdots \]
   converges absolutely if $p > 1$, converges conditionally if $0 < p \leq 1$, and diverges if $p \leq 0$.

4. As illustrated in the accompanying figure, a bug, starting at point $A$ on a 180 cm wire, walks the length of the wire, stops and walks in the opposite direction for half the length of the wire, stops again and walks in the opposite direction for one-third the length of the wire, stops again and walks in the opposite direction for one-fourth the length of the wire, and so forth until it stops for the 1000th time.
   (a) Give upper and lower bounds on the distance between the bug and point $A$ when it finally stops. [Hint: As stated in Example 2 of Section 9.6, assume that the sum of the alternating harmonic series is $\ln 2$.]
   (b) Give upper and lower bounds on the total distance that the bug has traveled when it finally stops. [Hint: Use inequality (2) of Section 9.4.]

   ![Figure Ex-4](image)

5. In Section 6.6 we defined the kinetic energy $K$ of a particle with mass $m$ and velocity $v$ to be $K = \frac{1}{2}mv^2$ [see Formula (7) of that section]. In this formula the mass $m$ is assumed to be constant, and $K$ is called the Newtonian kinetic energy. However, in Albert Einstein’s relativity theory the mass $m$ increases with the velocity and the kinetic energy $K$ is given by the formula
   \[ K = m_0c^2 \left[ \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right] \]
   in which $m_0$ is the mass of the particle when its velocity is zero, and $c$ is the speed of light. This is called the relativistic kinetic energy. Use an appropriate binomial series to show that if the velocity is small compared to the speed of light (i.e., $v/c \approx 0$), then the Newtonian and relativistic kinetic energies are in close agreement.

6. In Section 8.4 we studied the motion of a falling object that has mass $m$ and is retarded by air resistance. We showed that if the initial velocity is $v_0$ and the drag force $F_D$ is proportional to the velocity, that is, $F_D = -cv$, then the velocity of the object at time $t$ is
   \[ v(t) = e^{-ct/m} \left( v_0 + \frac{mg}{c} \right) - \frac{mg}{c} \]
   where $g$ is the acceleration due to gravity [see Formula (16) of Section 8.4].
   (a) Use a Maclaurin series to show that if $ct/m \approx 0$, then the velocity can be approximated as
   \[ v(t) \approx v_0 - \left( \frac{ct_0}{m} + g \right) t \]
   (b) Improve on the approximation in part (a).

EXPANDING THE CALCULUS HORIZON

To learn how ecologists use mathematical models based on the process of iteration to study the growth and decline of animal populations, see the module entitled Iteration and Dynamical Systems at:

www.wiley.com/college/anton
Mathematical curves, such as the spirals in the center of a sunflower, can be described conveniently using ideas developed in this chapter.

In this chapter we will study alternative ways of expressing curves in the plane. We will begin by studying parametric curves: curves described in terms of component functions. This study will include methods for finding tangent lines to parametric curves. We will then introduce polar coordinate systems and discuss methods for finding tangent lines to polar curves, arc length of polar curves, and areas enclosed by polar curves. Our attention will then turn to a review of the basic properties of conic sections: parabolas, ellipses, and hyperbolas. Finally, we will consider conic sections in the context of polar coordinates and discuss some applications in astronomy.

10.1 PARAMETRIC EQUATIONS; TANGENT LINES AND ARC LENGTH FOR PARAMETRIC CURVES

Graphs of functions must pass the vertical line test, a limitation that excludes curves with self-intersections or even such basic curves as circles. In this section we will study an alternative method for describing curves algebraically that is not subject to the severe restriction of the vertical line test. We will then derive formulas required to find slopes, tangent lines, and arc lengths of these parametric curves. We will conclude with an investigation of a classic parametric curve known as the cycloid.

PARAMETRIC EQUATIONS

Suppose that a particle moves along a curve \( C \) in the \( xy \)-plane in such a way that its \( x \)- and \( y \)-coordinates, as functions of time, are

\[
x = f(t), \quad y = g(t)
\]

We call these the parametric equations of motion for the particle and refer to \( C \) as the trajectory of the particle or the graph of the equations (Figure 10.1.1). The variable \( t \) is called the parameter for the equations.

**Example 1** Sketch the trajectory over the time interval \( 0 \leq t \leq 10 \) of the particle whose parametric equations of motion are

\[
x = t - 3\sin t, \quad y = 4 - 3\cos t
\]
10.1 Parametric Equations; Tangent Lines and Arc Length for Parametric Curves

Solution. One way to sketch the trajectory is to choose a representative succession of times, plot the \((x, y)\) coordinates of points on the trajectory at those times, and connect the points with a smooth curve. The trajectory in Figure 10.1.2 was obtained in this way from the data in Table 10.1.1 in which the approximate coordinates of the particle are given at time increments of 1 unit. Observe that there is no \(t\)-axis in the picture; the values of \(t\) appear only as labels on the plotted points, and even these are usually omitted unless it is important to emphasize the locations of the particle at specific times.

![Figure 10.1.2](image)

<table>
<thead>
<tr>
<th>(t)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>-1.5</td>
<td>2.4</td>
</tr>
<tr>
<td>2</td>
<td>-0.7</td>
<td>5.2</td>
</tr>
<tr>
<td>3</td>
<td>2.6</td>
<td>7.0</td>
</tr>
<tr>
<td>4</td>
<td>6.3</td>
<td>6.0</td>
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<td>7.9</td>
<td>3.1</td>
</tr>
<tr>
<td>6</td>
<td>6.8</td>
<td>1.1</td>
</tr>
<tr>
<td>7</td>
<td>5.0</td>
<td>1.7</td>
</tr>
<tr>
<td>8</td>
<td>5.0</td>
<td>4.4</td>
</tr>
<tr>
<td>9</td>
<td>7.8</td>
<td>6.7</td>
</tr>
<tr>
<td>10</td>
<td>11.6</td>
<td>6.5</td>
</tr>
</tbody>
</table>

Table 10.1.1

Although parametric equations commonly arise in problems of motion with time as the parameter, they arise in other contexts as well. Thus, unless the problem dictates that the parameter \(t\) in the equations \(x = f(t), \ y = g(t)\) represents time, it should be viewed simply as an independent variable that varies over some interval of real numbers. (In fact, there is no need to use the letter \(t\) for the parameter; any letter not reserved for another purpose can be used.) If no restrictions on the parameter are stated explicitly or implied by the equations, then it is understood that it varies from \(-\infty\) to \(+\infty\). To indicate that a parameter \(t\) is restricted to an interval \([a, b]\), we will write

\[
x = f(t), \quad y = g(t) \quad (a \leq t \leq b)
\]

Example 2 Find the graph of the parametric equations

\[
x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)
\]

Solution. One way to find the graph is to eliminate the parameter \(t\) by noting that

\[
x^2 + y^2 = \sin^2 t + \cos^2 t = 1
\]

Thus, the graph is contained in the unit circle \(x^2 + y^2 = 1\). Geometrically, the parameter \(t\) can be interpreted as the angle swept out by the radial line from the origin to the point \((x, y) = (\cos t, \sin t)\) on the unit circle (Figure 10.1.3). As \(t\) increases from 0 to \(2\pi\), the point traces the circle counterclockwise, starting at \((1, 0)\) when \(t = 0\) and completing one full revolution when \(t = 2\pi\). One can obtain different portions of the circle by varying the interval over which the parameter varies. For example,

\[
x = \cos t, \quad y = \sin t \quad (0 \leq t \leq \pi)
\]

represents just the upper semicircle in Figure 10.1.3.
ORIENTATION

The direction in which the graph of a pair of parametric equations is traced as the parameter increases is called the direction of increasing parameter or sometimes the orientation imposed on the curve by the equations. Thus, we make a distinction between a curve, which is a set of points, and a parametric curve, which is a curve with an orientation imposed on it by a set of parametric equations. For example, we saw in Example 2 that the circle represented parametrically by (2) is traced counterclockwise as \( t \) increases and hence has counterclockwise orientation. As shown in Figures 10.1.2 and 10.1.3, the orientation of a parametric curve can be indicated by arrowheads.

To obtain parametric equations for the unit circle with clockwise orientation, we can replace \( t \) by \(-t\) in (2) and use the identities \( \cos(-t) = \cos t \) and \( \sin(-t) = -\sin t \). This yields

\[
\begin{align*}
  x &= \cos t, \\
  y &= -\sin t \quad (0 \leq t \leq 2\pi)
\end{align*}
\]

Here, the circle is traced clockwise by a point that starts at \((1, 0)\) when \( t = 0 \) and completes one full revolution when \( t = 2\pi \) (Figure 10.1.4).

TECHNOLOGY MASTERY

When parametric equations are graphed using a calculator, the orientation can often be determined by watching the direction in which the graph is traced on the screen. However, many computers graph so fast that it is often hard to discern the orientation. See if you can use your graphing utility to confirm that (3) has a counterclockwise orientation.

Example 3

Graph the parametric curve

\[
\begin{align*}
  x &= 2t - 3, \\
  y &= 6t - 7
\end{align*}
\]

by eliminating the parameter, and indicate the orientation on the graph.

Solution. To eliminate the parameter we will solve the first equation for \( t \) as a function of \( x \), and then substitute this expression for \( t \) into the second equation:

\[
\begin{align*}
  t &= \left(\frac{1}{2}\right) (x + 3) \\
  y &= 6 \left(\frac{1}{2}\right) (x + 3) - 7 \\
  y &= 3x + 2
\end{align*}
\]

Thus, the graph is a line of slope 3 and \( y \)-intercept 2. To find the orientation we must look to the original equations; the direction of increasing \( t \) can be deduced by observing that \( x \) increases as \( t \) increases or by observing that \( y \) increases as \( t \) increases. Either piece of information tells us that the line is traced left to right as shown in Figure 10.1.5.

REMARK

Not all parametric equations produce curves with definite orientations; if the equations are badly behaved, then the point tracing the curve may leap around sporadically or move back and forth, failing to determine a definite direction. For example, if

\[
\begin{align*}
  x &= \sin t, \\
  y &= \sin^2 t
\end{align*}
\]

then the point \((x, y)\) moves along the parabola \( y = x^2 \). However, the value of \( x \) varies periodically between \(-1\) and 1, so the point \((x, y)\) moves periodically back and forth along the parabola between the points \((-1, 1)\) and \((1, 1)\) (as shown in Figure 10.1.6). Later in the text we will discuss restrictions that eliminate such erratic behavior, but for now we will just avoid such complications.

EXPRESSING ORDINARY FUNCTIONS PARAMETRICALLY

An equation \( y = f(x) \) can be expressed in parametric form by introducing the parameter \( t = x \); this yields the parametric equations

\[
\begin{align*}
  x &= t, \\
  y &= f(t)
\end{align*}
\]
10.1 Parametric Equations; Tangent Lines and Arc Length for Parametric Curves

For example, the portion of the curve \( y = \cos x \) over the interval \([-2\pi, 2\pi]\) can be expressed parametrically as 
\[
    x = t, \quad y = \cos t \quad (-2\pi \leq t \leq 2\pi)
\]
(Figure 10.1.7).

If a function \( f \) is one-to-one, then it has an inverse function \( f^{-1} \). In this case the equation 
\( y = f^{-1}(x) \) is equivalent to \( x = f(y) \). We can express the graph of \( f^{-1} \) in parametric form by introducing the parameter \( y = t \); this yields the parametric equations 
\[
    x = f(t), \quad y = t
\]
For example, Figure 10.1.8 shows the graph of \( f(x) = x^5 + x + 1 \) and its inverse. The graph of \( f \) can be represented parametrically as 
\[
    x = t, \quad y = t^5 + t + 1
\]
and the graph of \( f^{-1} \) can be represented parametrically as 
\[
    x = t^5 + t + 1, \quad y = t
\]

TANGENT LINES TO PARAMETRIC CURVES

We will be concerned with curves that are given by parametric equations
\[
    x = f(t), \quad y = g(t)
\]
in which \( f(t) \) and \( g(t) \) have continuous first derivatives with respect to \( t \). It can be proved that if \( dx/dt \neq 0 \), then \( y \) is a differentiable function of \( x \), in which case the chain rule implies that
\[
    \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (4)
\]
This formula makes it possible to find \( dy/dx \) directly from the parametric equations without eliminating the parameter.

**Example 4** Find the slope of the tangent line to the unit circle
\[
    x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)
\]
at the point where \( t = \pi/6 \) (Figure 10.1.9).

**Solution.** From (4), the slope at a general point on the circle is
\[
    \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t \quad (5)
\]
Thus, the slope at \( t = \pi/6 \) is
\[
    \left. \frac{dy}{dx} \right|_{t=\pi/6} = -\cot \frac{\pi}{6} = -\sqrt{3}
\]
It follows from Formula (4) that the tangent line to a parametric curve will be horizontal at those points where \( \frac{dy}{dt} = 0 \) and \( \frac{dx}{dt} \neq 0 \), since \( \frac{dy}{dx} = 0 \) at such points. Two different situations occur when \( \frac{dx}{dt} = 0 \). At points where \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} \neq 0 \), since \( \frac{dy}{dx} = 0 \) at such points. Two different situations occur when \( \frac{dx}{dt} = 0 \).

At points where \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} \neq 0 \), the right side of (4) has a nonzero numerator and a zero denominator; we will agree that the curve has infinite slope and a vertical tangent line at such points. At points where \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) are both zero, the right side of (4) becomes an indeterminate form; we call such points singular points. No general statement can be made about the behavior of parametric curves at singular points; they must be analyzed case by case.

Example 5

In a disastrous first flight, an experimental paper airplane follows the trajectory of the particle in Example 1:

\[
x = t - 3 \sin t, \quad y = 4 - 3 \cos t \quad (t \geq 0)
\]

but crashes into a wall at time \( t = 10 \) (Figure 10.1.11).

(a) At what times was the airplane flying horizontally?

(b) At what times was it flying vertically?

Solution (a). The airplane was flying horizontally at those times when \( \frac{dy}{dt} = 0 \) and \( \frac{dx}{dt} \neq 0 \). Setting \( \frac{dy}{dt} = 0 \) yields the equation

\[
3 \sin t = 0, \quad \text{or, more simply,} \quad \sin t = 0.
\]

This equation has four solutions in the time interval \( 0 \leq t \leq 10 \):

\[
t = 0, \quad t = \pi, \quad t = 2\pi, \quad t = 3\pi.
\]

Since \( \frac{dx}{dt} = 1 - 3 \cos t \neq 0 \) for these values of \( t \), the airplane was flying horizontally at times

\[
t = 0, \quad t = \pi \approx 3.14, \quad t = 2\pi \approx 6.28, \quad \text{and} \quad t = 3\pi \approx 9.42
\]

which is consistent with Figure 10.1.11.

Solution (b). The airplane was flying vertically at those times when \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} \neq 0 \). Setting \( \frac{dx}{dt} = 0 \) in (6) yields the equation

\[
1 - 3 \cos t = 0 \quad \text{or} \quad \cos t = \frac{1}{3}
\]

This equation has three solutions in the time interval \( 0 \leq t \leq 10 \) (Figure 10.1.12):

\[
t = \cos^{-1} \frac{1}{3}, \quad t = 2\pi - \cos^{-1} \frac{1}{3}, \quad t = 2\pi + \cos^{-1} \frac{1}{3}
\]
10.1 Parametric Equations; Tangent Lines and Arc Length for Parametric Curves

Since \( dy/dt = 3 \sin t \) is not zero at these points (why?), it follows that the airplane was flying vertically at times

\[
 t = \cos^{-1} \frac{1}{3} \approx 1.23, \quad t \approx 2\pi - 1.23 \approx 5.05, \quad t \approx 2\pi + 1.23 \approx 7.51
\]

which again is consistent with Figure 10.1.11. ◀

**Example 6** The curve represented by the parametric equations

\[
 x = t^2, \quad y = t^3 \quad (-\infty < t < +\infty)
\]

is called a **semicubical parabola**. The parameter \( t \) can be eliminated by cubing \( x \) and squaring \( y \), from which it follows that \( y^2 = x^3 \). The graph of this equation, shown in Figure 10.1.13, consists of two branches: an upper branch obtained by graphing \( y = x^{3/2} \) and a lower branch obtained by graphing \( y = -x^{3/2} \). The two branches meet at the origin, which corresponds to \( t = 0 \) in the parametric equations. This is a singular point because the derivatives \( dx/dt = 2t \) and \( dy/dt = 3t^2 \) are both zero there. ◀

**Example 7** Without eliminating the parameter, find \( dy/dx \) and \( d^2y/dx^2 \) at \((1, 1)\) and \((1, -1)\) on the semicubical parabola given by the parametric equations in Example 6.

**Solution.** From (4) we have

\[
 \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2t} = \frac{3}{2}, \quad (t \neq 0) \quad (7)
\]

and from (4) applied to \( y' = dy/dx \) we have

\[
 \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t}, \quad (8)
\]

Since the point \((1, 1)\) on the curve corresponds to \( t = 1 \) in the parametric equations, it follows from (7) and (8) that

\[
 \left. \frac{dy}{dx} \right|_{t=1} = \frac{3}{2} \quad \text{and} \quad \left. \frac{d^2y}{dx^2} \right|_{t=1} = \frac{3}{4}
\]

Similarly, the point \((1, -1)\) corresponds to \( t = -1 \) in the parametric equations, so applying (7) and (8) again yields

\[
 \left. \frac{dy}{dx} \right|_{t=-1} = \frac{3}{2} \quad \text{and} \quad \left. \frac{d^2y}{dx^2} \right|_{t=-1} = \frac{3}{4}
\]

Note that the values we obtained for the first and second derivatives are consistent with the graph in Figure 10.1.13, since at \((1, 1)\) on the upper branch the tangent line has positive slope and the curve is concave up, and at \((1, -1)\) on the lower branch the tangent line has negative slope and the curve is concave down.

Finally, observe that we were able to apply Formulas (7) and (8) for both \( t = 1 \) and \( t = -1 \), even though the points \((1, 1)\) and \((1, -1)\) lie on different branches. In contrast, had we chosen to perform the same computations by eliminating the parameter, we would have had to obtain separate derivative formulas for \( y = x^{3/2} \) and \( y = -x^{3/2} \). ◀

**Arc Length of Parametric Curves**

The following result provides a formula for finding the arc length of a curve from parametric equations for the curve. Its derivation is similar to that of Formula (3) in Section 6.4 and will be omitted.
Formulas (4) and (5) in Section 6.4 can be viewed as special cases of (9). For example, Formula (4) in Section 6.4 can be obtained from (9) by writing $y = f(x)$ parametrically as $x = t, \ y = f(t)$ and Formula (5) in Section 6.4 can be obtained by writing $x = g(y)$ parametrically as $x = g(t), \ y = t$.

**Example 8**

Use (9) to find the circumference of a circle of radius $a$ from the parametric equations

$x = a \cos t, \ y = a \sin t \quad (0 \leq t \leq 2\pi)$

**Solution.**

$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt = \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (a \cos t)^2} \ dt$

$= \int_0^{2\pi} a \ dt = a \left[ t \right]_0^{2\pi} = 2\pi a \quad \blacksquare$

**The Cycloid (The Apple of Discord)**

The results of this section can be used to investigate a curve known as a cycloid. This curve, which is one of the most significant in the history of mathematics, can be generated by a point on a circle that rolls along a straight line (Figure 10.1.14). This curve has a fascinating history, which we will discuss shortly; but first we will show how to obtain parametric equations for it. For this purpose, let us assume that the circle has radius $a$ and rolls along the positive $x$-axis of a rectangular coordinate system. Let $P(x, y)$ be the point on the circle that traces the cycloid, and assume that $P$ is initially at the origin. We will take as our parameter the angle $\theta$ that is swept out by the radial line to $P$ as the circle rolls (Figure 10.1.14). It is standard here to regard $\theta$ as positive, even though it is generated by a clockwise rotation.

The motion of $P$ is a combination of the movement of the circle’s center parallel to the $x$-axis and the rotation of $P$ about the center. As the radial line sweeps out an angle $\theta$, the point $P$ traverses an arc of length $a\theta$, and the circle moves a distance $a\theta$ along the $x$-axis. Thus, as suggested by Figure 10.1.15, the center moves to the point $(a\theta, a)$, and the coordinates of $P$ are

$x = a\theta - a \sin \theta, \ y = a - a \cos \theta \quad (10)$

These are the equations of the cycloid in terms of the parameter $\theta$.

One of the reasons the cycloid is important in the history of mathematics is that the study of its properties helped to spur the development of early versions of differentiation and integration. Work on the cycloid was carried out by some of the most famous names in seventeenth century mathematics, including Johann and Jakob Bernoulli, Descartes, L’Hôpital, Newton, and Leibniz. The curve was named the “cycloid” by the Italian mathematician and astronomer, Galileo, who spent over 40 years investigating its properties. An early problem of interest was that of constructing tangent lines to the cycloid. This problem was first solved by Descartes, and then by Fermat, whom Descartes had challenged with the question. A modern solution to this problem follows directly from the parametric equations (10) and Formula (4). For example, using Formula (4), it is straightforward to show that the $x$-intercepts of the cycloid are cusps and that there is a horizontal tangent line to the cycloid halfway between adjacent $x$-intercepts (Exercise 60).
Another early problem was determining the arc length of an arch of the cycloid. This was solved in 1658 by the famous British architect and mathematician, Sir Christopher Wren. He showed that the arc length of one arch of the cycloid is exactly eight times the radius of the generating circle. [For a solution to this problem using Formula (9), see Exercise 71.]

The cycloid is also important historically because it provides the solution to two famous mathematical problems—the brachistochrone problem (from Greek words meaning “shortest time”) and the tautochrone problem (from Greek words meaning “equal time”). The brachistochrone problem is to determine the shape of a wire along which a bead might slide from a point \( P \) to another point \( Q \), not directly below, in the shortest time. The tautochrone problem is to find the shape of a wire from \( P \) to \( Q \) such that two beads started at any points on the wire between \( P \) and \( Q \) reach \( Q \) in the same amount of time. The solution to both problems turns out to be an inverted cycloid (Figure 10.1.16).

In June of 1696, Johann Bernoulli posed the brachistochrone problem in the form of a challenge to other mathematicians. At first, one might conjecture that the wire should form a straight line, since that shape results in the shortest distance from \( P \) to \( Q \). However, the inverted cycloid allows the bead to fall more rapidly at first, building up sufficient speed to reach \( Q \) in the shortest time, even though it travels a longer distance. The problem was solved by Newton, Leibniz, and L'Hôpital, as well as by Johann Bernoulli and his older brother Jakob; it was formulated and solved incorrectly years earlier by Galileo, who thought the answer was a circular arc. In fact, Johann was so impressed with his brother Jakob's solution that he claimed it to be his own. (This was just one of many disputes about the cycloid that eventually led to the curve being known as the “apple of discord.”) One solution of the brachistochrone problem leads to the differential equation

\[
\left(1 + \left(\frac{dy}{dx}\right)^2\right)y = 2a
\]

where \( a \) is a positive constant. We leave it as an exercise (Exercise 72) to show that the cycloid provides a solution to this differential equation.
1. Find parametric equations for a circle of radius 2, centered at (3, 5).

2. The graph of the curve described by the parametric equations \( x = 4t - 1, y = 3t + 2 \) is a straight line with slope _____ and y-intercept _____.

3. Suppose that a parametric curve \( C \) is given by the equations \( x = f(t), y = g(t) \) for \( 0 \leq t \leq 1 \). Find parametric equations for \( C \) that reverse the direction the curve is traced as the parameter increases from 0 to 1.

4. To find \( dy/dx \) directly from the parametric equations
   \[
   x = f(t), \quad y = g(t)
   \]
   we can use the formula \( dy/dx = \)

5. Let \( L \) be the length of the curve
   \[
   x = \ln t, \quad y = \sin t \quad (1 \leq t \leq \pi)
   \]
   An integral expression for \( L \) is _____.

---

**Quick Check Exercises 10.1** (See page 705 for answers.)

1. (a) By eliminating the parameter, sketch the trajectory over the time interval \( 0 \leq t \leq 5 \) of the particle whose parametric equations of motion are
   \[
   x = t - 1, \quad y = t + 1
   \]

   (b) Indicate the direction of motion on your sketch.

   (c) Make a table of \( x \)- and \( y \)-coordinates of the particle at times \( t = 0, 1, 2, 3, 4, 5 \).

   (d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of \( t \).

---

**Johann (left) and Jakob (right) Bernoulli** Members of an amazing Swiss family that included several generations of outstanding mathematicians and scientists. Nikolaus Bernoulli (1623–1708), a druggist, fled from Antwerp to escape religious persecution and ultimately settled in Basel, Switzerland. There he had three sons, Jakob (also called Jean or James), Nikolaus, and Johann (also called Jacques or James). The Roman numerals are used to distinguish family members with identical names (see the family tree below).

Following Newton and Leibniz, the Bernoulli brothers, Jakob I and Johann I, are considered by some to be the two most important founders of calculus. Jakob I was self-taught in mathematics. His father wanted him to study for the ministry, but he turned to mathematics and in 1686 became a professor at the University of Basel. When he started working in mathematics, he knew nothing of Newton’s and Leibniz’ work. He eventually became familiar with Newton’s results, but because so little of Leibniz’ work was published, Jakob duplicated many of Leibniz’ results.

Jakob’s younger brother Johann I was urged to enter into business by his father. Instead, he turned to medicine and studied mathematics under the guidance of his older brother. He eventually became a mathematics professor at Gröningen in Holland, and then, when Jakob died in 1705, Johann succeeded him as mathematics professor at Basel. Throughout their lives, Jakob I and Johann I had a mutual passion for criticizing each other’s work, which frequently erupted into ugly confrontations. Leibniz tried to mediate the disputes, but Jakob, who resented Leibniz’ superior intellect, accused him of siding with Johann, and thus Leibniz became entangled in the arguments. The brothers often worked on common problems that they posed as challenges to one another. Johann, interested in gaining fame, often used unscrupulous means to make himself appear the originator of his brother’s results; Jakob occasionally retaliated. Thus, it is often difficult to determine who deserves credit for many results. However, both men made major contributions to the development of calculus. In addition to his work on calculus, Jakob helped establish fundamental principles in probability, including the Law of Large Numbers, which is a cornerstone of modern probability theory.

Among the other members of the Bernoulli family, Daniel, son of Johann I, is the most famous. He was a professor of mathematics at St. Petersburg Academy in Russia and subsequently a professor of anatomy and then physics at Basel. He did work in calculus and probability, but is best known for his work in physics. A basic law of fluid flow, called Bernoulli’s principle, is named in his honor. He won the annual prize of the French Academy 10 times for work on vibrating strings, tides of the sea, and kinetic theory of gases.

Johann II succeeded his father as professor of mathematics at Basel. His research was on the theory of heat and sound. Nikolaus I was a mathematician and law scholar who worked on probability and series. On the recommendation of Leibniz, he was appointed professor of mathematics at Padua and then went to Basel as a professor of logic and then law. Nikolaus II was professor of jurisprudence in Switzerland and then professor of mathematics at St. Petersburg Academy. Johann III was a professor of mathematics and astronomy in Berlin and Jakob II succeeded his uncle Daniel as professor of mathematics at St. Petersburg Academy in Russia. Truly an incredible family!
2. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \leq t \leq 1$ of the particle whose parametric equations of motion are

$$x = \cos(\pi t), \quad y = \sin(\pi t)$$

(b) Indicate the direction of motion on your sketch.

(c) Make a table of $x$- and $y$-coordinates of the particle at times $t = 0, 0.25, 0.5, 0.75, 1$.

(d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of $t$.

3–12 Sketch the curve by eliminating the parameter, and indicate the direction of increasing $t$.

3. $x = 3t - 4, \quad y = 6t + 2$
4. $x = t - 3, \quad y = 3t - 7 \quad (0 \leq t \leq 3)$
5. $x = 2 \cos t, \quad y = 5 \sin t \quad (0 \leq t \leq 2\pi)$
6. $x = \sqrt{t}, \quad y = 2t + 4$
7. $x = 3 + 2 \cos t, \quad y = 2 + 4 \sin t \quad (0 \leq t \leq 2\pi)$
8. $x = \sec t, \quad y = \tan t \quad (\pi < t < 3\pi/2)$
9. $x = \cos 2t, \quad y = \sin t \quad (-\pi/2 \leq t \leq \pi/2)$
10. $x = 4t + 3, \quad y = 16t^2 - 9$
11. $x = 2 \sin^2 t, \quad y = 3 \cos^2 t \quad (0 \leq t \leq \pi/2)$
12. $x = \sec^2 t, \quad y = \tan^2 t \quad (0 \leq t < \pi/2)$

13–18 Find parametric equations for the curve, and check your work by generating the curve with a graphing utility.

13. A circle of radius 5, centered at the origin, oriented clockwise.
14. The portion of the circle $x^2 + y^2 = 1$ that lies in the third quadrant, oriented counterclockwise.
15. A vertical line intersecting the $x$-axis at $x = 2$, oriented upward.
16. The ellipse $x^2/4 + y^2/9 = 1$, oriented counterclockwise.
17. The portion of the parabola $x = y^2$ joining $(1, -1)$ and $(1, 1)$, oriented down to up.
18. The circle of radius 4, centered at $(1, -3)$, oriented counterclockwise.

19. (a) Use a graphing utility to generate the trajectory of a particle whose equations of motion over the time interval $0 \leq t \leq 5$ are

$$x = 6t - \frac{1}{2}t^3, \quad y = 1 + \frac{1}{2}t^2$$

(b) Make a table of $x$- and $y$-coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.

(c) At what times is the particle on the $y$-axis?

(d) During what time interval is $y < 5$?

(e) At what time does the $x$-coordinate of the particle reach a maximum?

20. (a) Use a graphing utility to generate the trajectory of a paper airplane whose equations of motion for $t \geq 0$ are

$$x = t - 2 \sin t, \quad y = 3 - 2 \cos t$$

(b) Assuming that the plane flies in a room in which the floor is at $y = 0$, explain why the plane will not crash into the floor. [For simplicity, ignore the physical size of the plane by treating it as a particle.]

(c) How high must the ceiling be to ensure that the plane does not touch or crash into it?

21–22 Graph the equation using a graphing utility.

21. (a) $x = y^2 + 2y + 1$
(b) $x = \sin y, \quad -2\pi \leq y \leq 2\pi$

22. (a) $x = y + 2y^3 - y^5$
(b) $x = \tan y, \quad -\pi/2 < y < \pi/2$

23. In each part, match the parametric equation with one of the curves labeled (I)–(VI), and explain your reasoning.

(a) $x = \sqrt{t}, \quad y = \sin 3t \quad$ (b) $x = 2 \cos t, \quad y = 3 \sin t$
(c) $x = t \cos t, \quad y = t \sin t$
(d) $x = \frac{3t}{1 + t^2}, \quad y = \frac{3t^2}{1 + t^2}$
(e) $x = \frac{t^3}{1 + t^2}, \quad y = \frac{2t^2}{1 + t^2}$
(f) $x = \frac{1}{2} \cos t, \quad y = \sin 2t$

![Figure Ex-23]

24. (a) Identify the orientation of the curves in Exercise 23.
(b) Explain why the parametric curve

$$x = t^2, \quad y = t^4 \quad (-1 \leq t \leq 1)$$

does not have a definite orientation.

25. (a) Suppose that the line segment from the point $P(x_0, y_0)$ to $Q(x_1, y_1)$ is represented parametrically by

$$x = x_0 + (x_1 - x_0) t, \quad y = y_0 + (y_1 - y_0) t \quad (0 \leq t \leq 1)$$

and that $R(x, y)$ is the point on the line segment corresponding to a specified value of $t$ (see the accompanying figure on the next page). Show that $t = r/q$, where $r$ is the distance from $P$ to $R$ and $q$ is the distance from $P$ to $Q$.

(cont.)
(b) What value of $t$ produces the midpoint between points $P$ and $Q$?
(c) What value of $t$ produces the point that is three-fourths of the way from $P$ to $Q$?

26. Find parametric equations for the line segment joining $P(2, -1)$ and $Q(3, 1)$, and use the result in Exercise 25 to find
(a) the midpoint between $P$ and $Q$
(b) the point that is one-fourth of the way from $P$ to $Q$
(c) the point that is three-fourths of the way from $P$ to $Q$.

27. (a) Show that the line segment joining the points $(x_0, y_0)$ and $(x_1, y_1)$ can be represented parametrically as

\[ x = x_0 + (x_1 - x_0) \frac{t - t_0}{t_1 - t_0}, \quad (t_0 \leq t \leq t_1) \]

\[ y = y_0 + (y_1 - y_0) \frac{t - t_0}{t_1 - t_0}, \quad (t_0 \leq t \leq t_1) \]

(b) Which way is the line segment oriented?
(c) Find parametric equations for the line segment traced from $(3, -1)$ to $(1, 4)$ as $t$ varies from $1$ to $2$, and check your result with a graphing utility.

28. (a) By eliminating the parameter, show that if $a$ and $c$ are not both zero, then the graph of the parametric equations

\[ x = at + b, \quad y = ct + d \quad (t_0 \leq t \leq t_1) \]

is a line segment.
(b) Sketch the parametric curve

\[ x = 2t - 1, \quad y = t + 1 \quad (1 \leq t \leq 2) \]

and indicate its orientation.
(c) What can you say about the line in part (a) if $a$ or $c$ (but not both) is zero?
(d) What do the equations represent if $a$ and $c$ are both zero?

29–32 Use a graphing utility and parametric equations to display the graphs of $f$ and $f^{-1}$ on the same screen.

29. $f(x) = x^3 + 0.2x - 1, \quad -1 \leq x \leq 2$
30. $f(x) = \sqrt{x^2 + 2} + x, \quad -5 \leq x \leq 5$
31. $f(x) = \cos(0.5x), \quad 0 \leq x \leq 3$
32. $f(x) = x + \sin x, \quad 0 \leq x \leq 6$

33–36 True–False Determine whether the statement is true or false. Explain your answer.
33. The equation $y = 1 - x^2$ can be described parametrically by $x = \sin t, \ y = \cos^2 t$.
34. The graph of the parametric equations $x = f(t), \ y = t$ is the reflection of the graph of $y = f(x)$ about the $x$-axis.
35. For the parametric curve $x = x(t), \ y = 3t^4 - 2t^3$, the derivative of $y$ with respect to $x$ is computed by

\[ \frac{dy}{dx} = \frac{12t^3 - 6t^2}{x'(t)} \]

36. The curve represented by the parametric equations

\[ x = t^3, \quad y = t + t^6 \quad (-\infty < t < +\infty) \]

is concave down for $t < 0$.
37. Parametric curves can be defined piecewise by using different formulas for different values of the parameter. Sketch the curve that is represented piecewise by the parametric equations

\[ \begin{align*}
&x = 2t, \quad y = 4t^2 \quad (0 \leq t \leq \frac{1}{2}) \\
&x = 2 - 2t, \quad y = 2t \quad \left(\frac{1}{2} \leq t \leq 1\right)
\end{align*} \]

38. Find parametric equations for the rectangle in the accompanying figure, assuming that the rectangle is traced counterclockwise as $t$ varies from $0$ to $1$, starting at $(\frac{1}{2}, \frac{1}{2})$ when $t = 0$. [Hint: Represent the rectangle piecewise, letting $t$ vary from $0$ to $\frac{1}{2}$ for the first edge, from $\frac{1}{2}$ to $\frac{3}{2}$ for the second edge, and so forth.]

39. (a) Find parametric equations for the ellipse that is centered at the origin and has intercepts $(4, 0), (-4, 0), (0, 3),$ and $(0, -3)$.
(b) Find parametric equations for the ellipse that results by translating the ellipse in part (a) so that its center is at $(1, 2)$.
(c) Confirm your results in parts (a) and (b) using a graphing utility.
40. We will show later in the text that if a projectile is fired from ground level with an initial speed of $v_0$ meters per second at an angle $\alpha$ with the horizontal, and if air resistance is neglected, then its position after $t$ seconds, relative to the coordinate system in the accompanying figure on the next page is

\[ x = (v_0 \cos \alpha)t, \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \]

where $g \approx 9.8 \text{ m/s}^2$.
(a) By eliminating the parameter, show that the trajectory lies on the graph of a quadratic polynomial.
(b) Use a graphing utility to sketch the trajectory if $\alpha = 30^\circ$ and $v_0 = 1000 \text{ m/s}$.
(c) Using the trajectory in part (b), how high does the shell rise? (cont.)
In the mid-1850s the French physicist Jules Antoine Lissajous (1822–1880) became interested in parametric equations of the form
\[ x = \sin at, \quad y = \sin bt \]
in the course of studying vibrations that combine two perpendicular sinusoidal motions. If \( a/b \) is a rational number, then the combined effect of the oscillations is a periodic motion along a path called a Lissajous curve.

(a) Use a graphing utility to generate the complete graph of the Lissajous curves corresponding to \( a = 1, b = 2; a = 2, b = 3; a = 3, b = 4; \) and \( a = 4, b = 5 \).

(b) The Lissajous curve
\[ x = \sin t, \quad y = \sin 2t \quad (0 \leq t \leq 2\pi) \]
crosses itself at the origin (see Figure Ex-55). Find equations for the two tangent lines at the origin.

The prolate cycloid
\[ x = 2 - \pi \cos t, \quad y = 2\pi - \pi \sin t \quad (-\pi \leq t \leq \pi) \]
crosses itself at a point on the \( x \)-axis (see the accompanying figure). Find equations for the two tangent lines at that point.

57. Show that the curve \( x = t^2, \ y = t^3 - 4t \) intersects itself at the point \((4, 0)\), and find equations for the two tangent lines to the curve at the point of intersection.

58. Show that the curve with parametric equations
\[ x = t^2 - 3t + 5, \quad y = t^3 + r^2 - 10r + 9 \]
intersects itself at the point \((3, 1)\), and find equations for the two tangent lines to the curve at the point of intersection.

59. (a) Use a graphing utility to generate the graph of the parametric curve
\[ x = \cos^3 t, \quad y = \sin^3 t \quad (0 \leq t \leq 2\pi) \]
and make a conjecture about the values of \( t \) at which singular points occur.

(b) Confirm your conjecture in part (a) by calculating appropriate derivatives.

60. Verify that the cycloid described by Formula (10) has cusps at its \( x \)-intercepts and horizontal tangent lines at midpoints between adjacent \( x \)-intercepts (see Figure 10.1.14).
61. (a) What is the slope of the tangent line at time \( t \) to the trajectory of the paper airplane in Example 5? (b) What was the airplane’s approximate angle of inclination when it crashed into the wall?

62. Suppose that a bee follows the trajectory \( x = t - 2 \cos t, \quad y = 2 - 2 \sin t \quad (0 \leq t \leq 10) \)
   (a) At what times was the bee flying horizontally? (b) At what times was the bee flying vertically?

63. Consider the family of curves described by the parametric equations
   \[ x = a \cos t + h, \quad y = b \sin t + k \quad (0 \leq t < 2\pi) \]
   where \( a \neq 0 \) and \( b \neq 0 \). Describe the curves in this family if
   (a) \( h \) and \( k \) are fixed but \( a \) and \( b \) can vary
   (b) \( a \) and \( b \) are fixed but \( h \) and \( k \) can vary
   (c) \( a = 1 \) and \( b = 1 \), but \( h \) and \( k \) vary so that \( h = k + 1 \).

64. (a) Use a graphing utility to study how the curves in the family
   \[ x = 2a \cos^2 t, \quad y = 2a \cos t \sin t \quad (-2\pi < t < 2\pi) \]
   change as \( a \) varies from 0 to 5.
   (b) Confirm your conclusion algebraically.
   (c) Write a brief paragraph that describes your findings.

65–70 Find the exact arc length of the curve over the stated interval.

65. \( x = t^2, \quad y = \frac{1}{3}t^3 \quad (0 \leq t \leq 1) \)
66. \( x = \sqrt{t} - 2, \quad y = 2t^{3/4} \quad (1 \leq t \leq 16) \)
67. \( x = \cos 3t, \quad y = 3 \sin t \quad (0 \leq t \leq \pi) \)
68. \( x = \sin t + \cos t, \quad y = \sin t - \cos t \quad (0 \leq t \leq \pi) \)
69. \( x = e^t(\sin t + \cos t), \quad y = e^t(\sin t - \cos t) \quad (-1 \leq t \leq 1) \)
70. \( x = 2 \sin^{-1} t, \quad y = \ln(1 - t^2) \quad (0 \leq t \leq \frac{1}{2}) \)

71. (a) Use Formula (9) to show that the length \( L \) of one arch of a cycloid is given by
   \[ L = a \int_0^{2\pi} \sqrt{1 - \cos^2 \theta} \, d\theta \]
   (b) Use a CAS to show that \( L \) is eight times the radius of the wheel that generates the cycloid (see the accompanying figure).

72. Use the parametric equations in Formula (10) to verify that the cycloid provides one solution to the differential equation
   \[ \left( 1 + \left( \frac{dy}{dx} \right)^2 \right) y = 2a \]
   where \( a \) is a positive constant.

73. The amusement park rides illustrated in the accompanying figure consist of two connected rotating arms of length 1—an inner arm that rotates counterclockwise at 1 radian per second and an outer arm that can be programmed to rotate either clockwise at 2 radians per second (the Scrambler ride) or counterclockwise at 2 radians per second (the Calypso ride). The center of the rider cage is at the end of the outer arm.
   (a) Show that in the Scrambler ride the center of the cage has parametric equations
   \[ x = \cos t + \cos 2t, \quad y = \sin t - \sin 2t \]
   (b) Find parametric equations for the center of the cage in the Calypso ride, and use a graphing utility to confirm that the center traces the curve shown in the accompanying figure.
   (c) Do you think that a rider travels the same distance in one revolution of the Scrambler ride as in one revolution of the Calypso ride? Justify your conclusion.

74. (a) If a thread is unwound from a fixed circle while being held taut (i.e., tangent to the circle), then the end of the thread traces a curve called an involute of a circle. Show that if the circle is centered at the origin, has radius \( a \), and the end of the thread is initially at the point \((a, 0)\), then the involute can be expressed parametrically as
   \[ x = a(\cos \theta + \theta \sin \theta), \quad y = a(\sin \theta - \theta \cos \theta) \]
   where \( \theta \) is the angle shown in part (a) of the accompanying figure on the next page.
   (b) Assuming that the dog in part (b) of the accompanying figure on the next page unwinds its leash while keeping it taut, for what values of \( \theta \) in the interval \( 0 \leq \theta \leq 2\pi \) will the dog be walking North? South? East? West?
   (c) Use a graphing utility to generate the curve traced by the dog, and show that it is consistent with your answer in part (b).
10.2 Polar Coordinates

75–80 If \( f(t) \) and \( g'(t) \) are continuous functions, and if no segment of the curve
\[
    x = f(t), \quad y = g(t) \quad (a \leq t \leq b)
\]
is traced more than once, then it can be shown that the area of the surface generated by revolving this curve about the \( x \)-axis is
\[
    S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]
and the area of the surface generated by revolving the curve about the \( y \)-axis is
\[
    S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]
[The derivations are similar to those used to obtain Formulas (4) and (5) in Section 6.5.] Use the formulas above in these exercises.

75. Find the area of the surface generated by revolving \( x = t^2, \ y = 3t \ (0 \leq t \leq 2) \) about the \( x \)-axis.
76. Find the area of the surface generated by revolving the curve \( x = e^t \cos t, \ y = e^t \sin t \ (0 \leq t \leq \pi/2) \) about the \( x \)-axis.
77. Find the area of the surface generated by revolving the curve \( x = \cos^2 t, \ y = \sin^2 t \ (0 \leq t \leq \pi/2) \) about the \( y \)-axis.
78. Find the area of the surface generated by revolving \( x = 6t, \ y = 4t^2 \ (0 \leq t \leq 1) \) about the \( y \)-axis.
79. By revolving the semicircle \( x = r \cos t, \ y = r \sin t \ (0 \leq t \leq \pi) \) about the \( x \)-axis, show that the surface area of a sphere of radius \( r \) is \( 4\pi r^2 \).
80. The equations
\[
    x = a\phi - a \sin \phi, \quad y = a - a \cos \phi \quad (0 \leq \phi \leq 2\pi)
\]
represent one arch of a cycloid. Show that the surface area generated by revolving this curve about the \( x \)-axis is given by \( S = 64\pi a^2/3 \).
81. Writing Consult appropriate reference works and write an essay on American mathematician Nathaniel Bowditch (1773–1838) and his investigation of Bowditch curves (better known as Lissajous curves; see Exercise 55).
82. Writing What are some of the advantages of expressing a curve parametrically rather than in the form \( y = f(x) \)?
we can associate with each point \( P \) in the plane a pair of polar coordinates \((r, \theta)\), where \( r \) is the distance from \( P \) to the pole and \( \theta \) is an angle from the polar axis to the ray \( OP \) (Figure 10.2.1). The number \( r \) is called the radial coordinate of \( P \) and the number \( \theta \) the angular coordinate (or polar angle) of \( P \). In Figure 10.2.2, the points \((6, \pi/4)\), \((5, 2\pi/3)\), \((3, 5\pi/4)\), and \((4, 11\pi/6)\) are plotted in polar coordinate systems. If \( P \) is the pole, then \( r = 0 \), but there is no clearly defined polar angle. We will agree that an arbitrary angle can be used in this case; that is, \((0, \theta)\) are polar coordinates of the pole for all choices of \( \theta \).

The polar coordinates of a point are not unique. For example, the polar coordinates

\[
(1, 7\pi/4), \quad (1, -\pi/4), \quad \text{and} \quad (1, 15\pi/4)
\]

all represent the same point (Figure 10.2.3).

In general, if a point \( P \) has polar coordinates \((r, \theta)\), then

\[
(r, \theta + 2n\pi) \quad \text{and} \quad (r, \theta - 2n\pi)
\]

are also polar coordinates of \( P \) for any nonnegative integer \( n \). Thus, every point has infinitely many pairs of polar coordinates.

As defined above, the radial coordinate \( r \) of a point \( P \) is nonnegative, since it represents the distance from \( P \) to the pole. However, it will be convenient to allow for negative values of \( r \) as well. To motivate an appropriate definition, consider the point \( P \) with polar coordinates \((3, 5\pi/4)\). As shown in Figure 10.2.4, we can reach this point by rotating the polar axis through an angle of \( 5\pi/4 \) and then moving 3 units from the pole along the terminal side of the angle, or we can reach the point \( P \) by rotating the polar axis through an angle of \( \pi/4 \) and then moving 3 units from the pole along the extension of the terminal side. This suggests that the point \((3, 5\pi/4)\) might also be denoted by \((-3, \pi/4)\), with the minus sign serving to indicate that the point is on the extension of the angle’s terminal side rather than on the terminal side itself.

In general, the terminal side of the angle \( \theta + \pi \) is the extension of the terminal side of \( \theta \), so we define negative radial coordinates by agreeing that

\[
(-r, \theta) \quad \text{and} \quad (r, \theta + \pi)
\]

are polar coordinates of the same point.

**RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES**

Frequently, it will be useful to superimpose a rectangular \( xy \)-coordinate system on top of a polar coordinate system, making the positive \( x \)-axis coincide with the polar axis. If this is done, then every point \( P \) will have both rectangular coordinates \((x, y)\) and polar coordinates
10.2 Polar Coordinates

As suggested by Figure 10.2.5, these coordinates are related by the equations

\[ x = r \cos \theta, \quad y = r \sin \theta \]  

These equations are well suited for finding \( x \) and \( y \) when \( r \) and \( \theta \) are known. However, to find \( r \) and \( \theta \) when \( x \) and \( y \) are known, it is preferable to use the identities \( \sin^2 \theta + \cos^2 \theta = 1 \) and \( \tan \theta = \sin \theta / \cos \theta \) to rewrite (1) as

\[ r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x} \]  

Example 1 Find the rectangular coordinates of the point \( P \) whose polar coordinates are \((r, \theta) = (6, 2\pi/3)\) (Figure 10.2.6).

Solution. Substituting the polar coordinates \( r = 6 \) and \( \theta = 2\pi/3 \) in (1) yields

\[ x = 6 \cos \frac{2\pi}{3} = 6 \left(-\frac{1}{2}\right) = -3 \]
\[ y = 6 \sin \frac{2\pi}{3} = 6 \left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3} \]

Thus, the rectangular coordinates of \( P \) are \((x, y) = (-3, 3\sqrt{3})\).  

Example 2 Find polar coordinates of the point \( P \) whose rectangular coordinates are \((-2, -2\sqrt{3})\) (Figure 10.2.7).

Solution. We will find the polar coordinates \((r, \theta)\) of \( P \) that satisfy the conditions \( r > 0 \) and \( 0 \leq \theta < 2\pi \). From the first equation in (2),

\[ r^2 = x^2 + y^2 = (-2)^2 + (-2\sqrt{3})^2 = 4 + 12 = 16 \]

so \( r = 4 \). From the second equation in (2),

\[ \tan \theta = \frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \sqrt{3} \]

From this and the fact that \((-2, -2\sqrt{3})\) lies in the third quadrant, it follows that the angle satisfying the requirement \( 0 \leq \theta < 2\pi \) is \( \theta = 4\pi/3 \). Thus, \((r, \theta) = (4, 4\pi/3)\) are polar coordinates of \( P \). All other polar coordinates of \( P \) are expressible in the form

\[ \left(4, \frac{4\pi}{3} + 2n\pi\right) \quad \text{or} \quad \left(-4, \frac{\pi}{3} + 2n\pi\right) \]

where \( n \) is an integer.  

Graphs in Polar Coordinates

We will now consider the problem of graphing equations in \( r \) and \( \theta \), where \( \theta \) is assumed to be measured in radians. Some examples of such equations are

\[ r = 1, \quad \theta = \pi/4, \quad r = \theta, \quad r = \sin \theta, \quad r = \cos 2\theta \]

In a rectangular coordinate system the graph of an equation in \( x \) and \( y \) consists of all points whose coordinates \((x, y)\) satisfy the equation. However, in a polar coordinate system, points have infinitely many different pairs of polar coordinates, so that a given point may have some polar coordinates that satisfy an equation and others that do not. Given an equation
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in $r$ and $\theta$, we define its graph in polar coordinates to consist of all points with at least one pair of coordinates $(r, \theta)$ that satisfy the equation.

Example 3 Sketch the graphs of

(a) $r = 1$ (b) $\theta = \frac{\pi}{4}$

in polar coordinates.

Solution (a). For all values of $\theta$, the point $(1, \theta)$ is 1 unit away from the pole. Since $\theta$ is arbitrary, the graph is the circle of radius 1 centered at the pole (Figure 10.2.8a).

Solution (b). For all values of $r$, the point $(r, \pi/4)$ lies on a line that makes an angle of $\pi/4$ with the polar axis (Figure 10.2.8b). Positive values of $r$ correspond to points on the line in the first quadrant and negative values of $r$ to points on the line in the third quadrant. Thus, in absence of any restriction on $r$, the graph is the entire line. Observe, however, that had we imposed the restriction $r \geq 0$, the graph would have been just the ray in the first quadrant.

Figure 10.2.8

Equations $r = f(\theta)$ that express $r$ as a function of $\theta$ are especially important. One way to graph such an equation is to choose some typical values of $\theta$, calculate the corresponding values of $r$, and then plot the resulting pairs $(r, \theta)$ in a polar coordinate system. The next two examples illustrate this process.

Example 4 Sketch the graph of $r = \theta$ ($\theta \geq 0$) in polar coordinates by plotting points.

Solution. Observe that as $\theta$ increases, so does $r$; thus, the graph is a curve that spirals out from the pole as $\theta$ increases. A reasonably accurate sketch of the spiral can be obtained by plotting the points that correspond to values of $\theta$ that are integer multiples of $\pi/2$, keeping in mind that the value of $r$ is always equal to the value of $\theta$ (Figure 10.2.9).

Figure 10.2.9

Example 5 Sketch the graph of the equation $r = \sin \theta$ in polar coordinates by plotting points.

Solution. Table 10.2.1 shows the coordinates of points on the graph at increments of $\pi/6$.

These points are plotted in Figure 10.2.10. Note, however, that there are 13 points listed in the table but only 6 distinct plotted points. This is because the pairs from $\theta = \pi$ on yield
duplicates of the preceding points. For example, \((-1/2, 7\pi/6)\) and \((1/2, \pi/6)\) represent the same point.

### Table 10.2.1

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<th>(5\pi/6)</th>
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<td>(-\sqrt{3}/2)</td>
<td>(-1/2)</td>
<td>0</td>
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<tr>
<td>((r, \theta))</td>
<td>(0, 0)</td>
<td>((1/2, \pi/6))</td>
<td>((\sqrt{3}/2, \pi/3))</td>
<td>((1, \pi/2))</td>
<td>((\sqrt{3}/2, 2\pi/3))</td>
<td>((1/2, 5\pi/6))</td>
<td>((0, \pi))</td>
<td>((1, 7\pi/6))</td>
<td>((-\sqrt{3}/2, 4\pi/3))</td>
<td>((-1, 3\pi/2))</td>
<td>((-\sqrt{3}/2, 5\pi/3))</td>
<td>((-1/2, 11\pi/6))</td>
<td>((0, 2\pi))</td>
</tr>
</tbody>
</table>

Observe that the points in Figure 10.2.10 appear to lie on a circle. We can confirm that this is so by expressing the polar equation \(r = \sin \theta\) in terms of \(x\) and \(y\). To do this, we multiply the equation through by \(r\) to obtain

\[
r^2 = r \sin \theta
\]

which now allows us to apply Formulas (1) and (2) to rewrite the equation as

\[
x^2 + y^2 = y
\]

Rewriting this equation as \(x^2 + y^2 - y = 0\) and then completing the square yields

\[
x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}
\]

which is a circle of radius \(1/2\) centered at the point \((0, 1/2)\) in the \(xy\)-plane.

It is often useful to view the equation \(r = f(\theta)\) as an equation in rectangular coordinates (rather than polar coordinates) and graphed in a rectangular \(\theta r\)-coordinate system. For example, Figure 10.2.11 shows the graph of \(r = \sin \theta\) displayed using rectangular \(\theta r\)-coordinates. This graph can actually help to visualize how the polar graph in Figure 10.2.10 is generated:

- At \(\theta = 0\) we have \(r = 0\), which corresponds to the pole \((0, 0)\) on the polar graph.
- As \(\theta\) varies from 0 to \(\pi/2\), the value of \(r\) increases from 0 to 1, so the point \((r, \theta)\) moves along the circle from the pole to the high point at \((1, \pi/2)\).
- As \(\theta\) varies from \(\pi/2\) to \(\pi\), the value of \(r\) decreases from 1 back to 0, so the point \((r, \theta)\) moves along the circle from the high point back to the pole.
- As \(\theta\) varies from \(\pi\) to \(3\pi/2\), the values of \(r\) are negative, varying from 0 to \(-1\). Thus, the point \((r, \theta)\) moves along the circle from the pole to the high point at \((1, \pi/2)\), which is the same as the point \((-1, 3\pi/2)\). This duplicates the motion that occurred for \(0 \leq \theta \leq \pi/2\).
- As \(\theta\) varies from \(3\pi/2\) to \(2\pi\), the value of \(r\) varies from \(-1\) to 0. Thus, the point \((r, \theta)\) moves along the circle from the high point back to the pole, duplicating the motion that occurred for \(\pi/2 \leq \theta \leq \pi\).

#### Example 6

Sketch the graph of \(r = \cos 2\theta\) in polar coordinates.

**Solution.** Instead of plotting points, we will use the graph of \(r = \cos 2\theta\) in rectangular coordinates (Figure 10.2.12) to visualize how the polar graph of this equation is generated. The analysis and the resulting polar graph are shown in Figure 10.2.13. This curve is called a *four-petal rose*. ▶
Chapter 10 / Parametric and Polar Curves; Conic Sections

The converse of each part of Theorem 10.2.1 is false. See Exercise 79.

SYMMETRY TESTS

Observe that the polar graph of \( r = \cos 2\theta \) in Figure 10.2.13 is symmetric about the x-axis and the y-axis. This symmetry could have been predicted from the following theorem, which is suggested by Figure 10.2.14 (we omit the proof).

10.2.1 Theorem (Symmetry Tests)

(a) A curve in polar coordinates is symmetric about the x-axis if replacing \( \theta \) by \( -\theta \) in its equation produces an equivalent equation (Figure 10.2.14a).

(b) A curve in polar coordinates is symmetric about the y-axis if replacing \( \theta \) by \( \pi - \theta \) in its equation produces an equivalent equation (Figure 10.2.14b).

(c) A curve in polar coordinates is symmetric about the origin if replacing \( \theta \) by \( \theta + \pi \), or replacing \( r \) by \( -r \) in its equation produces an equivalent equation (Figure 10.2.14c).

Example 7 Use Theorem 10.2.1 to confirm that the graph of \( r = \cos 2\theta \) in Figure 10.2.13 is symmetric about the x-axis and y-axis.

Solution. To test for symmetry about the x-axis, we replace \( \theta \) by \( -\theta \). This yields

\[ r = \cos(-2\theta) = \cos 2\theta \]

Thus, replacing \( \theta \) by \( -\theta \) does not alter the equation.

To test for symmetry about the y-axis, we replace \( \theta \) by \( \pi - \theta \). This yields

\[ r = \cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = \cos(-2\theta) = \cos 2\theta \]

Thus, replacing \( \theta \) by \( \pi - \theta \) does not alter the equation.
10.2 Polar Coordinates

Example 8  Sketch the graph of \( r = a(1 - \cos \theta) \) in polar coordinates, assuming \( a \) to be a positive constant.

Solution. Observe first that replacing \( \theta \) by \( -\theta \) does not alter the equation, so we know in advance that the graph is symmetric about the polar axis. Thus, if we graph the upper half of the curve, then we can obtain the lower half by reflection about the polar axis.

As in our previous examples, we will first graph the equation in rectangular coordinates. This graph, which is shown in Figure 10.2.15a, can be obtained by rewriting the given equation as \( r = a - a \cos \theta \), from which we see that the graph in rectangular coordinates can be obtained by first reflecting the graph of \( r = a \cos \theta \) about the \( x \)-axis to obtain the graph of \( r = -a \cos \theta \), and then translating that graph up \( a \) units to obtain the graph of \( r = a - a \cos \theta \).

Now we can see the following:

- As \( \theta \) varies from 0 to \( \pi/3 \), \( r \) increases from 0 to \( a/2 \).
- As \( \theta \) varies from \( \pi/3 \) to \( \pi/2 \), \( r \) increases from \( a/2 \) to \( a \).
- As \( \theta \) varies from \( \pi/2 \) to \( (3\pi)/3 \), \( r \) increases from \( a \) to \( 3a/2 \).
- As \( \theta \) varies from \( (3\pi)/3 \) to \( \pi \), \( r \) increases from \( 3a/2 \) to \( 2a \).

This produces the polar curve shown in Figure 10.2.15b. The rest of the curve can be obtained by continuing the preceding analysis from \( \pi \) to \( 2\pi \) or, as noted above, by reflecting the portion already graphed about the \( x \)-axis (Figure 10.2.15c). This heart-shaped curve is called a cardioid (from the Greek word \( kardia \) meaning "heart").

Example 9  Sketch the graph of \( r^2 = 4 \cos 2\theta \) in polar coordinates.

Solution. This equation does not express \( r \) as a function of \( \theta \), since solving for \( r \) in terms of \( \theta \) yields two functions:

\[
r = 2\sqrt{\cos 2\theta} \quad \text{and} \quad r = -2\sqrt{\cos 2\theta}
\]

Thus, to graph the equation \( r^2 = 4 \cos 2\theta \) we will have to graph the two functions separately and then combine those graphs.

We will start with the graph of \( r = 2\sqrt{\cos 2\theta} \). Observe first that this equation is not changed if we replace \( \theta \) by \( -\theta \) or if we replace \( \theta \) by \( \pi - \theta \). Thus, the graph is symmetric about the \( x \)-axis and the \( y \)-axis. This means that the entire graph can be obtained by graphing the portion in the first quadrant, reflecting that portion about the \( y \)-axis to obtain the portion in the second quadrant, and then reflecting those two portions about the \( x \)-axis to obtain the portions in the third and fourth quadrants.
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To begin the analysis, we will graph the equation $r = 2\sqrt{\cos 2\theta}$ in rectangular $\theta r$-coordinates (see Figure 10.2.16a). Note that there are gaps in that graph over the intervals $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$ because $\cos 2\theta$ is negative for those values of $\theta$. From this graph we can see the following:

- As $\theta$ varies from 0 to $\pi/4$, $r$ decreases from 2 to 0.
- As $\theta$ varies from $\pi/4$ to $\pi/2$, no points are generated on the polar graph.

This produces the portion of the graph shown in Figure 10.2.16b. As noted above, we can complete the graph by a reflection about the $y$-axis followed by a reflection about the $x$-axis (Figure 10.2.16c). The resulting propeller-shaped graph is called a lemniscate (from the Greek word lemniscos for a looped ribbon resembling the number 8). We leave it for you to verify that the equation $r = -2\sqrt{\cos 2\theta}$, but traced in a diagonally opposite manner. Thus, the graph of the equation $r^2 = 4 \cos 2\theta$ consists of two identical superimposed lemniscates.

FAMILIES OF LINES AND RAYS THROUGH THE POLE

If $\theta_0$ is a fixed angle, then for all values of $r$ the point $(r, \theta_0)$ lies on the line that makes an angle of $\theta = \theta_0$ with the polar axis; and, conversely, every point on this line has a pair of polar coordinates of the form $(r, \theta_0)$. Thus, the equation $\theta = \theta_0$ represents the line that passes through the pole and makes an angle of $\theta_0$ with the polar axis (Figure 10.2.17a). If $r$ is restricted to be nonnegative, then the graph of the equation $\theta = \theta_0$ is the ray that emanates from the pole and makes an angle of $\theta_0$ with the polar axis (Figure 10.2.17b). Thus, as $\theta_0$ varies, the equation $\theta = \theta_0$ produces either a family of lines through the pole or a family of rays through the pole, depending on the restrictions on $r$.

FAMILIES OF CIRCLES

We will consider three families of circles in which $a$ is assumed to be a positive constant:

$$
\begin{align*}
r &= a \\
r &= 2a \cos \theta \\
r &= 2a \sin \theta
\end{align*}
$$

(3–5)

The equation $r = a$ represents a circle of radius $a$ centered at the pole (Figure 10.2.18a). Thus, as $a$ varies, this equation produces a family of circles centered at the pole. For families (4) and (5), recall from plane geometry that a triangle that is inscribed in a circle with a diameter of the circle for a side must be a right triangle. Thus, as indicated in Figures 10.2.18b and 10.2.18c, the equation $r = 2a \cos \theta$ represents a circle of radius $a$, centered on the $x$-axis and tangent to the $y$-axis at the origin; similarly, the equation $r = 2a \sin \theta$ represents a circle of radius $a$, centered on the $y$-axis and tangent to the $x$-axis at the origin. Thus, as $a$ varies, Equations (4) and (5) produce the families illustrated in Figures 10.2.18d and 10.2.18e.
10.2 Polar Coordinates

FAMILIES OF ROSE CURVES

In polar coordinates, equations of the form

\[ r = a \sin n\theta \quad \text{and} \quad r = a \cos n\theta \]

in which \( a > 0 \) and \( n \) is a positive integer represent families of flower-shaped curves called roses (Figure 10.2.19). The rose consists of \( n \) equally spaced petals of radius \( a \) if \( n \) is odd and \( 2n \) equally spaced petals of radius \( a \) if \( n \) is even. It can be shown that a rose with an even number of petals is traced out exactly once as \( \theta \) varies over the interval \( 0 \leq \theta < 2\pi \) and a rose with an odd number of petals is traced out exactly once as \( \theta \) varies over the interval \( 0 \leq \theta < \pi \) (Exercise 78). A four-petal rose of radius 1 was graphed in Example 6.

FAMILIES OF CARDIOIDS AND LIMAÇONS

Equations with any of the four forms

\[ r = a \pm b \sin \theta \quad \text{and} \quad r = a \pm b \cos \theta \]

in which \( a > 0 \) and \( b > 0 \) represent polar curves called limaçons (from the Latin word limax for a snail-like creature that is commonly called a “slug”). There are four possible shapes for a limaçon that are determined by the ratio \( a/b \) (Figure 10.2.20). If \( a = b \) (the case \( a/b = 1 \)), then the limaçon is called a cardioid because of its heart-shaped appearance, as noted in Example 8.
**Example 10** Figure 10.2.21 shows the family of limaçons $r = a + \cos \theta$ with the constant $a$ varying from 0.25 to 2.50 in steps of 0.25. In keeping with Figure 10.2.20, the limaçons evolve from the loop type to the convex type. As $a$ increases from the starting value of 0.25, the loops get smaller and smaller until the cardioid is reached at $a = 1$. As $a$ increases further, the limaçons evolve through the dimpled type into the convex type.

**FAMILIES OF SPIRALS**

A spiral is a curve that coils around a central point. Spirals generally have “left-hand” and “right-hand” versions that coil in opposite directions, depending on the restrictions on the polar angle and the signs of constants that appear in their equations. Some of the more common types of spirals are shown in Figure 10.2.22 for nonnegative values of $\theta$, $a$, and $b$.

**SPIRALS IN NATURE**

Spirals of many kinds occur in nature. For example, the shell of the chambered nautilus (below) forms a logarithmic spiral, and a coiled sailor’s rope forms an Archimedean spiral. Spirals also occur in flowers, the tusks of certain animals, and in the shapes of galaxies.
The shell of the chambered nautilus reveals a logarithmic spiral. The animal lives in the outermost chamber.

A sailor’s coiled rope forms an Archimedean spiral.

A spiral galaxy.

**GENERATING POLAR CURVES WITH GRAPHING UTILITIES**

For polar curves that are too complicated for hand computation, graphing utilities can be used. Although many graphing utilities are capable of graphing polar curves directly, some are not. However, if a graphing utility is capable of graphing parametric equations, then it can be used to graph a polar curve \( r = f(\theta) \) by converting this equation to parametric form. This can be done by substituting \( f(\theta) \) for \( r \) in (1). This yields

\[
\begin{align*}
x &= f(\theta) \cos \theta, \\
y &= f(\theta) \sin \theta
\end{align*}
\]

which is a pair of parametric equations for the polar curve in terms of the parameter \( \theta \).

**Example 11** Express the polar equation

\[ r = 2 + \cos \frac{5\theta}{2} \]

parametrically, and generate the polar graph from the parametric equations using a graphing utility.

**Solution.** Substituting the given expression for \( r \) in \( x = r \cos \theta \) and \( y = r \sin \theta \) yields the parametric equations

\[
\begin{align*}
x &= \left[ 2 + \cos \frac{5\theta}{2} \right] \cos \theta, \\
y &= \left[ 2 + \cos \frac{5\theta}{2} \right] \sin \theta
\end{align*}
\]

Next, we need to find an interval over which to vary \( \theta \) to produce the entire graph. To find such an interval, we will look for the smallest number of complete revolutions that must occur until the value of \( r \) begins to repeat. Algebraically, this amounts to finding the smallest positive integer \( n \) such that

\[
2 + \cos \left( \frac{5(\theta + 2n\pi)}{2} \right) = 2 + \cos \frac{5\theta}{2}
\]

or

\[
\cos \left( \frac{5\theta}{2} + 5n\pi \right) = \cos \frac{5\theta}{2}
\]
For this equality to hold, the quantity $5n\pi$ must be an even multiple of $\pi$; the smallest $n$ for which this occurs is $n = 2$. Thus, the entire graph will be traced in two revolutions, which means it can be generated from the parametric equations

$$x = \left[ 2 + \cos \frac{5\theta}{2} \right] \cos \theta, \quad y = \left[ 2 + \cos \frac{5\theta}{2} \right] \sin \theta \quad (0 \leq \theta \leq 4\pi)$$

This yields the graph in Figure 10.2.23.

\[\text{Figure 10.2.23}\]

**Quick Check Exercises 10.2**  (See page 719 for answers.)

1. (a) Rectangular coordinates of a point $(x, y)$ may be recovered from its polar coordinates $(r, \theta)$ by means of the equations $x = \ldots$ and $y = \ldots$.

(b) Polar coordinates $(r, \theta)$ may be recovered from rectangular coordinates $(x, y)$ by means of the equations $r^2 = \ldots$ and $\tan \theta = \ldots$

2. Find the rectangular coordinates of the points whose polar coordinates are given.

(a) $(4, \pi/3)$  (b) $(2, -\pi/6)$

(c) $(6, -2\pi/3)$  (d) $(4, 5\pi/4)$

3. In each part, find polar coordinates satisfying the stated conditions for the point whose rectangular coordinates are $(1, \sqrt{3})$.

(a) $r \geq 0$ and $0 \leq \theta < 2\pi$

(b) $r \leq 0$ and $0 \leq \theta < 2\pi$

4. In each part, state the name that describes the polar curve most precisely: a rose, a line, a circle, a limaçon, a cardioid, a spiral, a lemniscate, or none of these.

(a) $r = 1 - \theta$  (b) $r = 1 + 2\sin \theta$

(c) $r = \sin 2\theta$  (d) $r = \cos^2 \theta$

(e) $r = \csc \theta$  (f) $r = 2 + 2\cos \theta$

(g) $r = -2\sin \theta$

5. In each part, a point is given in rectangular coordinates. Find two pairs of polar coordinates for the point, one pair satisfying $r \geq 0$ and $0 \leq \theta < 2\pi$, and the second pair satisfying $r \geq 0$ and $-2\pi < \theta \leq 0$.

(a) $(-5, 0)$  (b) $(2\sqrt{3}, -2)$  (c) $(0, -2)$

(d) $(-8, -8)$  (e) $(-3, 3\sqrt{3})$  (f) $(1, 1)$

6. In each part, find polar coordinates satisfying the stated conditions for the point whose rectangular coordinates are $(-\sqrt{3}, 1)$.

(a) $r \geq 0$ and $0 \leq \theta < 2\pi$

(b) $r \leq 0$ and $0 \leq \theta < 2\pi$

(c) $r \geq 0$ and $-2\pi < \theta \leq 0$

(d) $r \leq 0$ and $-\pi < \theta \leq \pi$

7–8 Use a calculating utility, where needed, to approximate the polar coordinates of the points whose rectangular coordinates are given.

7. (a) $(3, 4)$  (b) $(-6, 8)$  (c) $(-1, \tan^{-1} 1)$

8. (a) $(-3, 4)$  (b) $(-3, 1.7)$  (c) $(2, \sin^{-1} \frac{1}{2})$

9–10 Identify the curve by transforming the given polar equation to rectangular coordinates.

9. (a) $r = 2$  (b) $r \sin \theta = 4$

(c) $r = 3 \cos \theta$  (d) $r = \frac{3\cos \theta + 2\sin \theta}{6}$

10. (a) $r = 5 \sec \theta$  (b) $r = 2 \sin \theta$

(c) $r = 4 \cos \theta + 4 \sin \theta$  (d) $r = \sec \theta \tan \theta$

11–12 Express the given equations in polar coordinates.

11. (a) $x = 3$  (b) $x^2 + y^2 = 7$

(c) $x^2 + y^2 + 6y = 0$  (d) $9xy = 4$
12. (a) \( y = -3 \)  
(b) \( x^2 + y^2 = 5 \)  
(c) \( x^2 + y^2 + 4x = 0 \)  
(d) \( x^2 + x^2 = y^2 \)

**FOCUS ON CONCEPTS**

13–16 A graph is given in a rectangular \( \theta r \)-coordinate system. Sketch the corresponding graph in polar coordinates.

13. 
14. 
15. 
16.

**17–20** Find an equation for the given polar graph. [Note: Numeric labels on these graphs represent distances to the origin.]

17. (a) Circle  
(b) Circle  
(c) Cardioid

18. (a) Limaçon  
(b) Circle  
(c) Cardioid

19. (a) Four-petal rose  
(b) Circle  
(c) Three-petal rose

20. (a) Cardioid  
(b) Limaçon  
(c) Lemniscate

**21–46** Sketch the curve in polar coordinates.  

21. \( \theta = \frac{\pi}{3} \)  
22. \( \theta = \frac{3\pi}{4} \)  
23. \( r = 3 \)  
24. \( r = 4 \cos \theta \)  
25. \( r = 6 \sin \theta \)  
26. \( r - 2 = 2 \cos \theta \)

**10.2 Polar Coordinates**

27. \( r = 3(1 + \sin \theta) \)
28. \( r = 5 - 5\sin \theta \)
29. \( r = 4 - 4\cos \theta \)
30. \( r = 1 + 2\sin \theta \)
31. \( r = -1 - \cos \theta \)
32. \( r = 4 + 3\cos \theta \)
33. \( r = 3 - \sin \theta \)
34. \( r = 3 + 4\cos \theta \)
35. \( r - 5 = 3\sin \theta \)
36. \( r = 5 - 2\cos \theta \)
37. \( r = -3 - 4\sin \theta \)
38. \( r^2 = \cos 2\theta \)
39. \( r^2 = 16 \sin 2\theta \)
40. \( r = 4\theta \) \( (\theta \geq 0) \)
41. \( r = 4\theta \) \( (\theta \leq 0) \)
42. \( r = 4\theta \)
43. \( r = -2 \cos 2\theta \)
44. \( r = 3 \sin 2\theta \)
45. \( r = 9 \sin 4\theta \)
46. \( r = 2 \cos 3\theta \)

**47–50** True–False Determine whether the statement is true or false. Explain your answer.

47. The polar coordinate pairs \((-1, \pi/3)\) and \((1, -2\pi/3)\) describe the same point.
48. If the graph of \( r = f(\theta) \) drawn in rectangular \( \theta r \)-coordinates is symmetric about the \( r \)-axis, then the graph of \( r = f(\theta) \) drawn in polar coordinates is symmetric about the \( x \)-axis.
49. The portion of the polar graph of \( r = \sin 2\theta \) for values of \( \theta \) between \( \pi/2 \) and \( \pi \) is contained in the second quadrant.
50. The graph of a dimpled limaçon passes through the polar origin.

**51–55** Determine a shortest parameter interval on which a complete graph of the polar equation can be generated, and then use a graphing utility to generate the polar graph.

51. \( r = \cos \frac{\theta}{2} \)
52. \( r = \sin \frac{\theta}{2} \)
53. \( r = 1 - 2 \sin \frac{\theta}{4} \)
54. \( r = 0.5 + \cos \frac{\theta}{3} \)
55. \( r = \cos \frac{\theta}{5} \)

56. The accompanying figure shows the graph of the “butterfly curve” 

\[ r = e^{\cos \theta} - 2 \cos 4\theta + \sin^3 \frac{\theta}{4} \]

Determine a shortest parameter interval on which the complete butterfly can be generated, and then check your answer using a graphing utility.

\[ \text{Figure Ex-56} \]
57. The accompanying figure shows the Archimedean spiral \( r = \theta / 2 \) produced with a graphing calculator.
(a) What interval of values for \( \theta \) do you think was used to generate the graph?
(b) Duplicate the graph with your own graphing utility.

58. Find equations for the two families of circles in the accompanying figure.

59. (a) Show that if \( a \) varies, then the polar equation \( r = a \csc \theta \ (0 < \theta < \pi) \)
describes a family of lines parallel to the polar axis.
(b) Show that if \( b \) varies, then the polar equation \( r = b \sec \theta \ (\pi/2 < \theta < \pi) \)
describes a family of lines perpendicular to the polar axis.

60. The accompanying figure shows graphs of the Archimedean spiral \( r = \theta \) and the parabolic spiral \( r = \sqrt{\theta} \).
Which is which? Explain your reasoning.

61–62 A polar graph of \( r = f(\theta) \) is given over the stated interval. Sketch the graph of
(a) \( r = f(-\theta) \)
(b) \( r = f(\theta - \pi/2) \)
(c) \( r = f(\theta + \pi/2) \)
(d) \( r = -f(\theta) \).

63–64 Use the polar graph from the indicated exercise to sketch the graph of
(a) \( r = f(\theta) + 1 \)
(b) \( r = 2f(\theta) - 1 \).

65. Show that if the polar graph of \( r = f(\theta) \) is rotated counterclockwise around the origin through an angle \( \alpha \), then \( r = f(\theta - \alpha) \) is an equation for the rotated curve. [Hint: If \((r_0, \theta_0)\) is any point on the original graph, then \((r_0, \theta_0 + \alpha)\) is a point on the rotated graph.]

66. Use the result in Exercise 65 to find an equation for the lemniscate that results when the lemniscate in Example 9 is rotated counterclockwise through an angle of \( \pi/2 \).

67. Use the result in Exercise 65 to find an equation for the cardioid \( r = 1 + \cos \theta \) after it has been rotated through the given angle, and check your answer with a graphing utility.
(a) \( \pi/4 \)
(b) \( \pi/2 \)
(c) \( \pi \)
(d) \( 5\pi/4 \)

68. (a) Show that if \( A \) and \( B \) are not both zero, then the graph of the polar equation
\[ r = A \sin \theta + B \cos \theta \]
is a circle. Find its radius.
(b) Derive Formulas (4) and (5) from the formula given in part (a).

69. Find the highest point on the cardioid \( r = 1 + \cos \theta \).
70. Find the leftmost point on the upper half of the cardioid \( r = 1 + \cos \theta \).

71. Show that in a polar coordinate system the distance \( d \) between the points \((r_1, \theta_1)\) and \((r_2, \theta_2)\) is
\[ d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)} \]

72–74 Use the formula obtained in Exercise 71 to find the distance between the two points indicated in polar coordinates.
(a) \((3, \pi/6)\) and \((2, \pi/3)\)
73. Successive tips of the four-petal rose \( r = \cos 2\theta \). Check your answer using geometry.

74. Successive tips of the three-petal rose \( r = \sin 3\theta \). Check your answer using trigonometry.

75. In the late seventeenth century the Italian astronomer Giovanni Domenico Cassini (1625–1712) introduced the family of curves
\[
(x^2 + y^2 + a^2)^2 - b^4 - 4a^2x^2 = 0 \quad (a > 0, b > 0)
\]
in his studies of the relative motions of the Earth and the Sun. These curves, which are called Cassini ovals, have one of the three basic shapes shown in the accompanying figure.

(a) Show that if \( a = b \), then the polar equation of the Cassini oval is \( r^2 = 2a^2 \cos 2\theta \), which is a lemniscate.

(b) Use the formula in Exercise 71 to show that the lemniscate in part (a) is the curve traced by a point that moves in such a way that the product of its distances from the polar points \((a, 0)\) and \((a, \pi)\) is \(a^2\).

![Figure Ex-75](image)

76–77 Vertical and horizontal asymptotes of polar curves can sometimes be detected by investigating the behavior of \( x = r \cos \theta \) and \( y = r \sin \theta \) as \( \theta \) varies. This idea is used in these exercises. ■

✓ QUICK CHECK ANSWERS 10.2

1. (a) \( r \cos \theta \); \( r \sin \theta \) \hspace{1cm} (b) \( x^2 + y^2 \); \( y/x \)

2. (a) \((2, 2\sqrt{3})\) \hspace{1cm} (b) \((-\sqrt{3}, -1)\) \hspace{1cm} (c) \((-3, -3\sqrt{3})\) \hspace{1cm} (d) \((-2\sqrt{2}, -2\sqrt{2})\)

3. (a) \((2, \pi/3)\) \hspace{1cm} (b) \((-2, 4\pi/3)\)

4. (a) spiral \hspace{1cm} (b) limaçon \hspace{1cm} (c) rose \hspace{1cm} (d) none of these \hspace{1cm} (e) line \hspace{1cm} (f) cardioid \hspace{1cm} (g) circle

10.3 Tangent Lines, Arc Length, and Area for Polar Curves

In this section we will derive the formulas required to find slopes, tangent lines, and arc lengths of polar curves. We will then show how to find areas of regions that are bounded by polar curves.

TANGENT LINES TO POLAR CURVES

Our first objective in this section is to find a method for obtaining slopes of tangent lines to polar curves of the form \( r = f(\theta) \) in which \( r \) is a differentiable function of \( \theta \). We showed in the last section that a curve of this form can be expressed parametrically in terms of the parameter \( \theta \) by substituting \( f(\theta) \) for \( r \) in the equations \( x = r \cos \theta \) and \( y = r \sin \theta \). This yields
\[
x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta
\]
from which we obtain
\[
\frac{dx}{d\theta} = -f(\theta) \sin \theta + f'(\theta) \cos \theta = -r \sin \theta + \frac{dr}{d\theta} \cos \theta
\]
\[
\frac{dy}{d\theta} = f(\theta) \cos \theta + f'(\theta) \sin \theta = r \cos \theta + \frac{dr}{d\theta} \sin \theta
\]
(1)

Thus, if \( \frac{dx}{d\theta} \) and \( \frac{dy}{d\theta} \) are continuous and if \( \frac{dx}{d\theta} \neq 0 \), then \( y \) is a differentiable function of \( x \), and Formula (4) in Section 10.1 with \( \theta \) in place of \( t \) yields
\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}}
\]
(2)

**Example 1** Find the slope of the tangent line to the circle \( r = 4 \cos \theta \) at the point where \( \theta = \pi/4 \).

**Solution.** From (2) with \( r = 4 \cos \theta \), so that \( dr/d\theta = -4 \sin \theta \), we obtain
\[
\frac{dy}{dx} = \frac{4 \cos^2 \theta - 4 \sin^2 \theta}{-8 \sin \theta \cos \theta} = -\frac{\cos 2\theta}{2 \sin \theta \cos \theta}
\]
Using the double-angle formulas for sine and cosine,
\[
\frac{dy}{dx} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta
\]
Thus, at the point where \( \theta = \pi/4 \) the slope of the tangent line is
\[
m = \frac{dy}{dx}\bigg|_{\theta=\pi/4} = -\cot \frac{\pi}{2} = 0
\]
which implies that the circle has a horizontal tangent line at the point where \( \theta = \pi/4 \) (Figure 10.3.1).

**Example 2** Find the points on the cardioid \( r = 1 - \cos \theta \) at which there is a horizontal tangent line, a vertical tangent line, or a singular point.

**Solution.** A horizontal tangent line will occur where \( \frac{dy}{d\theta} = 0 \) and \( \frac{dx}{d\theta} \neq 0 \), a vertical tangent line where \( \frac{dy}{d\theta} \neq 0 \) and \( \frac{dx}{d\theta} = 0 \), and a singular point where \( \frac{dy}{d\theta} = 0 \) and \( \frac{dx}{d\theta} = 0 \). We could find these derivatives from the formulas in (1). However, an alternative approach is to go back to basic principles and express the cardioid parametrically by substituting \( r = 1 - \cos \theta \) in the conversion formulas \( x = r \cos \theta \) and \( y = r \sin \theta \). This yields
\[
x = (1 - \cos \theta) \cos \theta, \quad y = (1 - \cos \theta) \sin \theta \quad (0 \leq \theta \leq 2\pi)
\]
Differentiating these equations with respect to \( \theta \) and then simplifying yields (verify)
\[
\frac{dx}{d\theta} = \sin \theta (2 \cos \theta - 1), \quad \frac{dy}{d\theta} = (1 - \cos \theta)(1 + 2 \cos \theta)
\]
Thus, \( dx/d\theta = 0 \) if \( \sin \theta = 0 \) or \( \cos \theta = \frac{1}{2} \), and \( dy/d\theta = 0 \) if \( \cos \theta = 1 \) or \( \cos \theta = -\frac{1}{2} \). We leave it for you to solve these equations and show that the solutions of \( dx/d\theta = 0 \) on the interval \( 0 \leq \theta \leq 2\pi \) are
\[
dx{d\theta} = 0: \quad \theta = 0, \quad \frac{\pi}{3}, \quad \pi, \quad \frac{5\pi}{3}, \quad 2\pi
\]
10.3 Tangent Lines, Arc Length, and Area for Polar Curves

and the solutions of \( dy/d\theta = 0 \) on the interval \( 0 \leq \theta \leq 2\pi \) are

\[
\frac{dy}{d\theta} = 0: \quad \theta = 0, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad 2\pi
\]

Thus, horizontal tangent lines occur at \( \theta = \frac{2\pi}{3} \) and \( \theta = \frac{4\pi}{3} \); vertical tangent lines occur at \( \theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{3} \); and singular points occur at \( \theta = 0 \) and \( \theta = 2\pi \) (Figure 10.3.2).

Note, however, that \( r = 0 \) at both singular points, so there is really only one singular point on the cardioid—the pole.

**TANGENT LINES TO POLAR CURVES AT THE ORIGIN**

Formula (2) reveals some useful information about the behavior of a polar curve \( r = f(\theta) \) that passes through the origin. If we assume that \( r = 0 \) and \( dr/d\theta \neq 0 \) when \( \theta = \theta_0 \), then it follows from Formula (2) that the slope of the tangent line to the curve at \( \theta = \theta_0 \) is

\[
\tan \theta_0
\]

(Figure 10.3.3). However, \( \tan \theta_0 \) is also the slope of the line \( \theta = \theta_0 \), so we can conclude that this line is tangent to the curve at the origin. Thus, we have established the following result.

**10.3.1 THEOREM** If the polar curve \( r = f(\theta) \) passes through the origin at \( \theta = \theta_0 \), and if \( dr/d\theta \neq 0 \) at \( \theta = \theta_0 \), then the line \( \theta = \theta_0 \) is tangent to the curve at the origin.

This theorem tells us that equations of the tangent lines at the origin to the curve \( r = f(\theta) \) can be obtained by solving the equation \( f(\theta) = 0 \). It is important to keep in mind, however, that \( r = f(\theta) \) may be zero for more than one value of \( \theta \), so there may be more than one tangent line at the origin. This is illustrated in the next example.

**Example 3** The three-petal rose \( r = \sin 3\theta \) in Figure 10.3.4 has three tangent lines at the origin, which can be found by solving the equation

\[
\sin 3\theta = 0
\]

It was shown in Exercise 78 of Section 10.2 that the complete rose is traced once as \( \theta \) varies over the interval \( 0 \leq \theta < \pi \), so we need only look for solutions in this interval. We leave it for you to confirm that these solutions are

\[
\theta = 0, \quad \theta = \frac{\pi}{3}, \quad \text{and} \quad \theta = \frac{2\pi}{3}
\]

Since \( dr/d\theta = 3 \cos 3\theta \neq 0 \) for these values of \( \theta \), these three lines are tangent to the rose at the origin, which is consistent with the figure.

**ARC LENGTH OF A POLAR CURVE**

A formula for the arc length of a polar curve \( r = f(\theta) \) can be derived by expressing the curve in parametric form and applying Formula (9) of Section 10.1 for the arc length of a parametric curve. We leave it as an exercise to show the following.
10.3.2 **Arc Length Formula for Polar Curves** If no segment of the polar curve \( r = f(\theta) \) is traced more than once as \( \theta \) increases from \( \alpha \) to \( \beta \), and if \( \frac{dr}{d\theta} \) is continuous for \( \alpha \leq \theta \leq \beta \), then the arc length \( L \) from \( \theta = \alpha \) to \( \theta = \beta \) is

\[
L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta
\]  

(3)

**Example 4** Find the arc length of the spiral \( r = e^\theta \) in Figure 10.3.5 between \( \theta = 0 \) and \( \theta = \pi \).

**Solution.**

\[
L = \int_{0}^{\pi} \sqrt{e^{2\theta} + (e^\theta)^2} \, d\theta = \sqrt{2} \left. e^{\theta} \right|_{0}^{\pi} = \sqrt{2}(e^{\pi} - 1) \approx 31.3
\]

**Example 5** Find the total arc length of the cardioid \( r = 1 + \cos \theta \).

**Solution.** The cardioid is traced out once as \( \theta \) varies from \( \theta = 0 \) to \( \theta = 2\pi \). Thus,

\[
L = \int_{0}^{2\pi} \sqrt{1 + \cos^2 \theta} \, d\theta = \sqrt{2} \int_{0}^{2\pi} \sqrt{1 + \cos \theta} \, d\theta = 2 \int_{0}^{2\pi} \sqrt{\cos^2 \frac{\theta}{2}} \, d\theta
\]

Identity (45) of Appendix B

Since \( \cos \frac{\theta}{2} \) changes sign at \( \pi \), we must split the last integral into the sum of two integrals: the integral from 0 to \( \pi \) plus the integral from \( \pi \) to \( 2\pi \). However, the integral from \( \pi \) to \( 2\pi \) is equal to the integral from 0 to \( \pi \), since the cardioid is symmetric about the polar axis (Figure 10.3.6). Thus,

\[
L = 2 \int_{0}^{\pi} \cos \frac{\theta}{2} \, d\theta = 8 \sin \frac{\pi}{2} \left. \right|_{0}^{\pi/2} = 8
\]

**Area in Polar Coordinates**

We begin our investigation of area in polar coordinates with a simple case.

10.3.3 **Area Problem in Polar Coordinates** Suppose that \( \alpha \) and \( \beta \) are angles that satisfy the condition \( \alpha < \beta \leq \alpha + 2\pi \) and suppose that \( f(\theta) \) is continuous and nonnegative for \( \alpha \leq \theta \leq \beta \). Find the area of the region \( R \) enclosed by the polar curve \( r = f(\theta) \) and the rays \( \theta = \alpha \) and \( \theta = \beta \) (Figure 10.3.7).
In rectangular coordinates we obtained areas under curves by dividing the region into an increasing number of vertical strips, approximating the strips by rectangles, and taking a limit. In polar coordinates rectangles are clumsy to work with, and it is better to partition the region into wedges by using rays \( \theta = \theta_1, \theta = \theta_2, \ldots, \theta = \theta_{n-1} \) such that \( \alpha < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \beta \) (Figure 10.3.8). As shown in that figure, the rays divide the region \( R \) into \( n \) wedges with areas \( A_1, A_2, \ldots, A_n \) and central angles \( \Delta \theta_1, \Delta \theta_2, \ldots, \Delta \theta_n \). The area of the entire region can be written as

\[
A = A_1 + A_2 + \cdots + A_n = \sum_{k=1}^{n} A_k \tag{4}
\]

If \( \Delta \theta_k \) is small, then we can approximate the area \( A_k \) of the \( k \)th wedge by the area of a sector with central angle \( \Delta \theta_k \) and radius \( f(\theta^*_k) \), where \( \theta^*_k \) is any ray that lies in the \( k \)th wedge (Figure 10.3.9). Thus, from (4) and Formula (5) of Appendix B for the area of a sector, we obtain

\[
A = \sum_{k=1}^{n} A_k \approx \sum_{k=1}^{n} \frac{1}{2} [f(\theta^*_k)]^2 \Delta \theta_k \tag{5}
\]

If we now increase \( n \) in such a way that \( \max \Delta \theta_k \to 0 \), then the sectors will become better and better approximations of the wedges and it is reasonable to expect that (5) will approach the exact value of the area \( A \) (Figure 10.3.10); that is,

\[
A = \lim_{\max \Delta \theta_k \to 0} \sum_{k=1}^{n} \frac{1}{2} [f(\theta^*_k)]^2 \Delta \theta_k = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 \, d\theta
\]

Note that the discussion above can easily be adapted to the case where \( f(\theta) \) is nonpositive for \( \alpha \leq \theta \leq \beta \). We summarize this result below.

10.3.4 AREA IN POLAR COORDINATES

If \( \alpha \) and \( \beta \) are angles that satisfy the condition

\[
\alpha < \beta \leq \alpha + 2\pi
\]

and if \( f(\theta) \) is continuous and either nonnegative or nonpositive for \( \alpha \leq \theta \leq \beta \), then the area \( A \) of the region \( R \) enclosed by the polar curve \( r = f(\theta) \) \((\alpha \leq \theta \leq \beta)\) and the lines \( \theta = \alpha \) and \( \theta = \beta \) is

\[
A = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \tag{6}
\]

The hardest part of applying (6) is determining the limits of integration. This can be done as follows:

Area in Polar Coordinates: Limits of Integration

Step 1. Sketch the region \( R \) whose area is to be determined.

Step 2. Draw an arbitrary “radial line” from the pole to the boundary curve \( r = f(\theta) \).

Step 3. Ask, “Over what interval of values must \( \theta \) vary in order for the radial line to sweep out the region \( R \)?”

Step 4. Your answer in Step 3 will determine the lower and upper limits of integration.
Example 6  

Find the area of the region in the first quadrant that is within the cardioid \( r = 1 - \cos \theta \).

Solution.  

The region and a typical radial line are shown in Figure 10.3.11. For the radial line to sweep out the region, \( \theta \) must vary from 0 to \( \pi/2 \). Thus, from (6) with \( \alpha = 0 \) and \( \beta = \pi/2 \), we obtain

\[
A = \frac{1}{2} \int_{\pi/2}^{0} (1 - \cos \theta)^2 d\theta = \frac{1}{2} \int_{0}^{\pi/2} (1 - \cos \theta)(1 - \cos \theta) d\theta
\]

With the help of the identity \( \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) \), this can be rewritten as

\[
A = \frac{1}{2} \int_{\pi/2}^{0} \left( \frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta = \frac{1}{2} \left[ \frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} = \frac{3}{8} \pi - 1
\]

Example 7  

Find the entire area within the cardioid of Example 6.

Solution.  

For the radial line to sweep out the entire cardioid, \( \theta \) must vary from 0 to \( 2\pi \). Thus, from (6) with \( \alpha = 0 \) and \( \beta = 2\pi \),

\[
A = \frac{1}{2} \int_{2\pi}^{0} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{0}^{2\pi} (1 - \cos \theta)^2 d\theta
\]

If we proceed as in Example 6, this reduces to

\[
A = \frac{1}{2} \int_{0}^{2\pi} \left( \frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta = \frac{3\pi}{2}
\]

Alternative Solution.  

Since the cardioid is symmetric about the \( x \)-axis, we can calculate the portion of the area above the \( x \)-axis and double the result. In the portion of the cardioid above the \( x \)-axis, \( \theta \) ranges from 0 to \( \pi \), so that

\[
A = 2 \int_{0}^{\pi} \frac{1}{2} r^2 d\theta = \int_{0}^{\pi} (1 - \cos \theta)^2 d\theta = \frac{3\pi}{2}
\]

Using Symmetry  

Although Formula (6) is applicable if \( r = f(\theta) \) is negative, area computations can sometimes be simplified by using symmetry to restrict the limits of integration to intervals where \( r \geq 0 \). This is illustrated in the next example.

Example 8  

Find the area of the region enclosed by the rose curve \( r = \cos 2\theta \).

Solution.  

Referring to Figure 10.2.13 and using symmetry, the area in the first quadrant that is swept out for \( 0 \leq \theta \leq \pi/4 \) is one-eighth of the total area inside the rose. Thus, from Formula (6)

\[
A = 8 \int_{0}^{\pi/8} \frac{1}{2} r^2 d\theta = 4 \int_{0}^{\pi/4} \cos^2 2\theta d\theta
\]

\[
= 4 \int_{0}^{\pi/4} \frac{1}{2} (1 + \cos 4\theta) d\theta = 2 \int_{0}^{\pi/4} (1 + \cos 4\theta) d\theta
\]

\[
= 2\theta + \frac{1}{2} \sin 4\theta \bigg|_{0}^{\pi/4} = \frac{\pi}{2}
\]
10.3 Tangent Lines, Arc Length, and Area for Polar Curves

Sometimes the most natural way to satisfy the restriction \( \alpha < \beta \leq \alpha + 2\pi \) required by Formula (6) is to use a negative value for \( \alpha \). For example, suppose that we are interested in finding the area of the shaded region in Figure 10.3.12a. The first step would be to determine the intersections of the cardioid \( r = 4 + 4 \cos \theta \) and the circle \( r = 6 \), since this information is needed for the limits of integration. To find the points of intersection, we can equate the two expressions for \( r \). This yields

\[
4 + 4 \cos \theta = 6 \quad \text{or} \quad \cos \theta = \frac{1}{2}
\]

which is satisfied by the positive angles

\[
\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}
\]

However, there is a problem here because the radial lines to the circle and cardioid do not sweep through the shaded region shown in Figure 10.3.12b as \( \theta \) varies over the interval \( \pi/3 \leq \theta \leq 5\pi/3 \). There are two ways to circumvent this problem—one is to take advantage of the symmetry by integrating over the interval \( 0 \leq \theta \leq \pi/3 \) and doubling the result, and the second is to use a negative lower limit of integration and integrate over the interval \( -\pi/3 \leq \theta \leq \pi/3 \) (Figure 10.3.12c). The two methods are illustrated in the next example.

**Example 9**  
Find the area of the region that is inside of the cardioid \( r = 4 + 4 \cos \theta \) and outside of the circle \( r = 6 \).

**Solution Using a Negative Angle.**  
The area of the region can be obtained by subtracting the areas in Figures 10.3.12d and 10.3.12e:

\[
A = \int_{-\pi/3}^{\pi/3} \frac{1}{2} (4 + 4 \cos \theta)^2 \, d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2} (6)^2 \, d\theta
\]

\[
= \int_{-\pi/3}^{\pi/3} \frac{1}{2} [(4 + 4 \cos \theta)^2 - 36] \, d\theta = \int_{-\pi/3}^{\pi/3} (16 \cos \theta + 8 \cos^2 \theta - 10) \, d\theta
\]

\[
= \left[ 16 \sin \theta + (4 \theta + 2 \sin 2\theta) - 10 \theta \right]_{-\pi/3}^{\pi/3} = 18\sqrt{3} - 4\pi
\]

**Solution Using Symmetry.**  
Using symmetry, we can calculate the area above the polar axis and double it. This yields (verify)

\[
A = 2 \int_{0}^{\pi/3} \frac{1}{2} [(4 + 4 \cos \theta)^2 - 36] \, d\theta = 2(9\sqrt{3} - 2\pi) = 18\sqrt{3} - 4\pi
\]

which agrees with the preceding result.
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**INTERSECTIONS OF POLAR GRAPHS**

In the last example we found the intersections of the cardioid and circle by equating their expressions for \( r \) and solving for \( \theta \). However, because a point can be represented in different ways in polar coordinates, this procedure will not always produce all of the intersections. For example, the cardioids

\[
 r = 1 - \cos \theta \quad \text{and} \quad r = 1 + \cos \theta
\]

intersect at three points: the pole, the point \((1, \pi/2)\), and the point \((1, 3\pi/2)\) (Figure 10.3.13). Equating the right-hand sides of the equations in (7) yields \( 1 - \cos \theta = 1 + \cos \theta \) or \( \cos \theta = 0 \), so

\[
 \theta = \frac{\pi}{2} + k\pi, \quad k = 0, \pm 1, \pm 2, \ldots
\]

Substituting any of these values in (7) yields \( r = 1 \), so that we have found only two distinct points of intersection, \((1, \pi/2)\) and \((1, 3\pi/2)\); the pole has been missed. This problem occurs because the two cardioids pass through the pole at different values of \( \theta \)—the cardioid \( r = 1 - \cos \theta \) passes through the pole at \( \theta = 0 \), and the cardioid \( r = 1 + \cos \theta \) passes through the pole at \( \theta = \pi \).

The situation with the cardioids is analogous to two satellites circling the Earth in intersecting orbits (Figure 10.3.14). The satellites will not collide unless they reach the same point at the same time. In general, when looking for intersections of polar curves, it is a good idea to graph the curves to determine how many intersections there should be.

**QUICK CHECK EXERCISES 10.3** *(See page 729 for answers.)*

1. (a) To obtain \( dy/dx \) directly from the polar equation
   \[ r = f(\theta), \]
   we can use the formula
   \[
   \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \cdots
   \]
   (b) Use the formula in part (a) to find \( dy/dx \) directly from the polar equation \( r = \csc \theta \).

2. (a) What conditions on \( f(\theta_0) \) and \( f'(\theta_0) \) guarantee that the line \( \theta = \theta_0 \) is tangent to the polar curve \( r = f(\theta) \) at the origin?
   (b) What are the values of \( \theta_0 \) in \([0, 2\pi]\) at which the lines \( \theta = \theta_0 \) are tangent at the origin to the four-petal rose \( r = \cos 2\theta \)?

3. (a) To find the arc length \( L \) of the polar curve \( r = f(\theta) \) \((\alpha \leq \theta \leq \beta)\), we can use the formula \( L = \cdots \).
   (b) The polar curve \( r = \sec \theta \) \((0 \leq \theta \leq \pi/4)\) has arc length \( L = \cdots \).

4. The area of the region enclosed by a nonnegative polar curve \( r = f(\theta) \) \((\alpha \leq \theta \leq \beta)\) and the lines \( \theta = \alpha \) and \( \theta = \beta \) is given by the definite integral \( \cdots \).

5. Find the area of the circle \( r = a \) by integration.

**EXERCISE SET 10.3**  

**Graphing Utility** **CAS**

1–6 Find the slope of the tangent line to the polar curve for the given value of \( \theta \).

1. \( r = 2 \sin \theta; \quad \theta = \pi/6 \)
2. \( r = 1 + \cos \theta; \quad \theta = \pi/2 \)
3. \( r = 1/\theta; \quad \theta = 2 \)
4. \( r = a \sec 2\theta; \quad \theta = \pi/6 \)
5. \( r = \sin 3\theta; \quad \theta = \pi/4 \)
6. \( r = 4 - 3 \sin \theta; \quad \theta = \pi \)

7–8 Calculate the slopes of the tangent lines indicated in the accompanying figures.

7. \( r = 2 + 2 \sin \theta \)
8. \( r = 1 - 2 \sin \theta \)

9–10 Find polar coordinates of all points at which the polar curve has a horizontal or a vertical tangent line.

9. \( r = a(1 + \cos \theta) \)
10. \( r = a \sin \theta \)
11–12 Use a graphing utility to make a conjecture about the number of points on the polar curve at which there is a horizontal tangent line, and confirm your conjecture by finding appropriate derivatives.

11. \( r = \sin \theta \cos^2 \theta \)
12. \( r = 1 - 2 \sin \theta \)

13–18 Sketch the polar curve and find polar equations of the tangent lines to the curve at the pole.

13. \( r = 2 \cos 3\theta \)
14. \( r = 4 \sin \theta \)
15. \( r = 4 \cos 2\theta \)
16. \( r = \sin 2\theta \)
17. \( r = 1 - 2 \cos \theta \)
18. \( r = 2 \theta \)

19–22 Use Formula (3) to calculate the arc length of the polar curve.

19. The entire circle \( r = a \)
20. The entire circle \( r = 2a \cos \theta \)
21. The entire cardioid \( r = a(1 - \cos \theta) \)
22. \( r = e^{3\theta} \) from \( \theta = 0 \) to \( \theta = 2 \)
23. (a) Show that the arc length of one petal of the rose \( r = \cos n\theta \) is given by

\[
2 \int_0^{\pi/(2n)} \sqrt{1 + (n^2 - 1) \sin^2 n\theta} \, d\theta
\]

(b) Use the numerical integration capability of a calculating utility to approximate the arc length of one petal of the four-petal rose \( r = \cos 2\theta \).
(c) Use the numerical integration capability of a calculating utility to approximate the arc length of one petal of the \( n \)-petal rose \( r = \cos n\theta \) for \( n = 2, 3, 4, \ldots, 20 \); then make a conjecture about the limit of these arc lengths as \( n \to +\infty \).

24. (a) Sketch the spiral \( r = e^{-\theta/8} \) (\( 0 \leq \theta < +\infty \)).
(b) Find an improper integral for the total arc length of the spiral.
(c) Show that the integral converges and find the total arc length of the spiral.

25. Write down, but do not evaluate, an integral for the area of each shaded region.

26. Find the area of the shaded region in Exercise 25(d).

27. In each part, find the area of the circle by integration.
(a) \( r = 2a \sin \theta \)
(b) \( r = 2a \cos \theta \)

28. (a) Show that \( r = 2 \sin \theta + 2 \cos \theta \) is a circle.
(b) Find the area of the circle using a geometric formula and then by integration.

29–34 Find the area of the region described.

29. The region that is enclosed by the cardioid \( r = 2 + 2 \sin \theta \).
30. The region in the first quadrant within the cardioid \( r = 1 + \cos \theta \).
31. The region enclosed by the rose \( r = 4 \cos 3\theta \).
32. The region enclosed by the rose \( r = 2 \sin 2\theta \).
33. The region enclosed by the inner loop of the limaçon \( r = 1 + 2 \cos \theta \). [Hint: \( r \leq 0 \) over the interval of integration.]
34. The region swept out by a radial line from the pole to the curve \( r = 2/\theta \) as \( \theta \) varies over the interval \( 1 \leq \theta \leq 3 \).

35–38 Find the area of the shaded region.

35. \[
\begin{array}{c}
\text{t = cos} \\
\text{t = 2 cos} \\
\text{t = sin} \\
\end{array}
\]
36. \[
\begin{array}{c}
\text{t = cos} \\
\text{t = 4 \cos} \\
\end{array}
\]
37. \[
\begin{array}{c}
\text{t = cos} \\
\text{t = 4 \cos} \\
\end{array}
\]
38. \[
\begin{array}{c}
\text{t = 1 + cos} \\
\text{t = 3 cos} \\
\end{array}
\]

39–46 Find the area of the region described.

39. The region inside the circle \( r = 3 \sin \theta \) and outside the cardioid \( r = 1 + \sin \theta \).
40. The region outside the cardioid \( r = 2 - 2 \cos \theta \) and inside the circle \( r = 4 \).
41. The region inside the cardioid \( r = 2 + 2 \cos \theta \) and outside the circle \( r = 3 \).
42. The region that is common to the circles \( r = 2 \cos \theta \) and \( r = 2 \sin \theta \).
43. The region between the loops of the limaçon \( r = \frac{1}{2} + \cos \theta \).
44. The region inside the cardioid \( r = 2 + 2 \cos \theta \) and to the right of the line \( r \cos \theta = \frac{1}{2} \).
45. The region inside the circle \( r = 2 \) and to the right of the line \( r = \sqrt{2} \sec \theta \).
46. The region inside the rose \( r = 2a \cos 2\theta \) and outside the circle \( r = a \sqrt{2} \).
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47–50 True–False Determine whether the statement is true or false. Explain your answer.

47. The x-axis is tangent to the polar curve \( r = \cos(\theta/2) \) at \( \theta = 3\pi \).

48. The arc length of the polar curve \( r = \sqrt{\theta} \) for \( 0 \leq \theta \leq \pi/2 \) is given by

\[
L = \int_{0}^{\pi/2} \sqrt{1 + \frac{1}{4\theta^2}} \, d\theta
\]

49. The area of a sector with central angle \( \theta \) taken from a circle of radius \( r \) is \( \theta r^2 \).

50. The expression

\[
\frac{1}{2} \int_{-\pi/4}^{\pi/4} (1 - \sqrt{2} \cos \theta)^2 \, d\theta
\]

computes the area enclosed by the inner loop of the limaçon \( r = 1 - \sqrt{2} \cos \theta \).

**Focus on Concepts**

51. (a) Find the error: The area that is inside the lemniscate \( r^2 = a^2 \cos 2\theta \) is

\[
A = \int_{0}^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_{0}^{2\pi} \frac{1}{2} a^2 \cos 2\theta \, d\theta
\]

\[
= \frac{1}{2} a^2 \sin 2\theta \bigg|_{0}^{2\pi} = 0
\]

(b) Find the correct area.

(c) Find the area inside the lemniscate \( r^2 = 4 \cos 2\theta \) and outside the circle \( r = \sqrt{2} \).

52. Find the area inside the curve \( r^2 = \sin 2\theta \).

53. A radial line is drawn from the origin to the spiral \( r = a\theta \) \((a > 0 \text{ and } \theta \geq 0)\). Find the area swept out during the second revolution of the radial line that was not swept out during the first revolution.

54. As illustrated in the accompanying figure, suppose that a rod with one end fixed at the pole of a polar coordinate system rotates counterclockwise at the constant rate of 1 rad/s. At time \( t = 0 \) a bug on the rod is 10 mm from the pole and is moving outward along the rod at the constant speed of 2 mm/s.

(a) Find an equation of the form \( r = f(\theta) \) for the path of motion of the bug, assuming that \( \theta = 0 \) when \( t = 0 \).

(b) Find the distance the bug travels along the path in part (a) during the first 5 s. Round your answer to the nearest tenth of a millimeter.

55. (a) Show that the Folium of Descartes \( x^3 - 3xy + y^3 = 0 \) can be expressed in polar coordinates as

\[
r = \frac{3 \sin \theta \cos \theta}{\cos^3 \theta + \sin^3 \theta}
\]

(b) Use a CAS to show that the area inside the loop is \( \frac{3}{2} \) (Figure 3.1.3a).

56. (a) What is the area that is enclosed by one petal of the rose \( r = a \cos n\theta \) if \( n \) is an even integer?

(b) What is the area that is enclosed by one petal of the rose \( r = a \cos n\theta \) if \( n \) is an odd integer?

(c) Use a CAS to show that the total area enclosed by the rose \( r = a \cos n\theta \) is \( \pi a^2/2 \) if the number of petals is even. [Hint: See Exercise 78 of Section 10.2.]

(d) Use a CAS to show that the total area enclosed by the rose \( r = a \cos n\theta \) is \( \pi a^2/4 \) if the number of petals is odd.

57. One of the most famous problems in Greek antiquity was “squaring the circle,” that is, using a straightedge and compass to construct a square whose area is equal to that of a given circle. It was proved in the nineteenth century that no such construction is possible. However, show that the shaded areas in the accompanying figure are equal, thereby “squaring the crescent.”

58. Use a graphing utility to generate the polar graph of the equation \( r = \cos 3\theta + 2 \), and find the area that it encloses.

59. Use a graphing utility to generate the graph of the bifolium

\[
r = 2 \cos \theta \sin^2 \theta
\]

and find the area of the upper loop.

60. Use Formula (9) of Section 10.1 to derive the arc length formula for polar curves, Formula (3).

61. As illustrated in the accompanying figure, let \( P(r, \theta) \) be a point on the polar curve \( r = f(\theta) \), let \( \psi \) be the smallest counterclockwise angle from the extended radius \( OP \) to the
10.3 Tangent Lines, Arc Length, and Area for Polar Curves

62–63 Use the formula for \( \psi \) obtained in Exercise 61. ■

62. (a) Use the trigonometric identity

\[
\tan \theta = \frac{1 - \cos \theta}{\sin \theta}
\]

to show that if \((r, \theta)\) is a point on the cardioid

\[
r = 1 - \cos \theta \quad (0 \leq \theta < 2\pi)
\]

then \( \psi = \theta/2 \).

(b) Sketch the cardioid and show the angle \( \psi \) at the points where the cardioid crosses the \( y \)-axis.

(c) Find the angle \( \psi \) at the points where the cardioid crosses the \( y \)-axis.

63. Show that for a logarithmic spiral \( r = ae^{b\theta} \), the angle from the radial line to the tangent line is constant along the spiral (see the accompanying figure). [Note: For this reason, logarithmic spirals are sometimes called *equiangular spirals*.]

64. (a) In the discussion associated with Exercises 75–80 of Section 10.1, formulas were given for the area of the surface of revolution that is generated by revolving a parametric curve about the \( x \)-axis or \( y \)-axis. Use those formulas to derive the following formulas for the areas of the surfaces of revolution that are generated by revolving the portion of the polar curve \( r = f(\theta) \) from \( \theta = \alpha \) to \( \theta = \beta \) about the polar axis and about the line \( \theta = \pi/2 \):

\[
S = \int_a^\beta \frac{1}{2} f(\theta)^2 \, d\theta \quad \text{About } \theta = \alpha
\]

\[
S = \int_a^\beta \frac{1}{2} (r^2 + \frac{dr}{d\theta}^2) \, d\theta \quad \text{About } \theta = \pi/2
\]

(b) State conditions under which these formulas hold.

65–68 Sketch the surface, and use the formulas in Exercise 64 to find the surface area. ■

65. The surface generated by revolving the circle \( r = \cos \theta \) about the line \( \theta = \pi/2 \).

66. The surface generated by revolving the spiral \( r = e^{\theta} \) \((0 \leq \theta \leq \pi/2) \) about the line \( \theta = \pi/2 \).

67. The “apple” generated by revolving the upper half of the cardioid \( r = 1 - \cos \theta \) \((0 \leq \theta \leq \pi) \) about the polar axis.

68. The sphere of radius \( a \) generated by revolving the semicircle \( r = a \) in the upper half-plane about the polar axis.

69. **Writing**

(a) Show that if \( 0 \leq \theta_1 < \theta_2 \leq \pi \) and if \( r_1 \) and \( r_2 \) are positive, then the area \( A \) of a triangle with vertices \((0,0)\), \((r_1, \theta_1)\), and \((r_2, \theta_2)\) is

\[
A = \frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1)
\]

(b) Use the formula obtained in part (a) to describe an approach to answer Area Problem 10.3.3 that uses an approximation of the region \( R \) by triangles instead of circular wedges. Reconcile your approach with Formula (6).

70. **Writing** In order to find the area of a region bounded by two polar curves it is often necessary to determine their points of intersection. Give an example to illustrate that the points of intersection of curves \( r = f(\theta) \) and \( r = g(\theta) \) may not coincide with solutions to \( f(\theta) = g(\theta) \). Discuss some strategies for determining intersection points of polar curves and provide examples to illustrate your strategies.

---

**QUICK CHECK ANSWERS 10.3**

1. (a) \( \frac{dr}{d\theta} \) (b) \( dy = 0 \)

2. (a) \( f(\theta_0) = 0 \), \( f'(\theta_0) \neq 0 \) (b) \( \theta_0 = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \)

3. (a) \( \int_a^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2} \, d\theta \) (b) \( 1 \)

4. \( \int_a^\beta \frac{1}{2} f(\theta)^2 \, d\theta = \int_a^\beta \frac{1}{2} r^2 \, d\theta \)

5. \( \int_0^{2\pi} \frac{1}{2} a^2 \, d\theta = \pi a^2 \)
In this section we will discuss some of the basic geometric properties of parabolas, ellipses, and hyperbolas. These curves play an important role in calculus and also arise naturally in a broad range of applications in such fields as planetary motion, design of telescopes and antennas, geodetic positioning, and medicine, to name a few.

CONIC SECTIONS

Circles, ellipses, parabolas, and hyperbolas are called *conic sections* or *conics* because they can be obtained as intersections of a plane with a double-napped circular cone (Figure 10.4.1). If the plane passes through the vertex of the double-napped cone, then the intersection is a point, a pair of intersecting lines, or a single line. These are called *degenerate conic sections*.

![Figure 10.4.1](image)

Some students may already be familiar with the material in this section, in which case it can be treated as a review. Instructors who want to spend some additional time on precalculus review may want to allocate more than one lecture on this material.
10.4 Conic Sections

DEFINITIONS OF THE CONIC SECTIONS

Although we could derive properties of parabolas, ellipses, and hyperbolas by defining them as intersections with a double-napped cone, it will be better suited to calculus if we begin with equivalent definitions that are based on their geometric properties.

10.4.1 Definition A parabola is the set of all points in the plane that are equidistant from a fixed line and a fixed point not on the line.

The line is called the directrix of the parabola, and the point is called the focus (Figure 10.4.2). A parabola is symmetric about the line that passes through the focus at right angles to the directrix. This line, called the axis or the axis of symmetry of the parabola, intersects the parabola at a point called the vertex.

10.4.2 Definition An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points is a given positive constant that is greater than the distance between the fixed points.

The two fixed points are called the foci (plural of “focus”) of the ellipse, and the midpoint of the line segment joining the foci is called the center (Figure 10.4.3a). To help visualize Definition 10.4.2, imagine that two ends of a string are tacked to the foci and a pencil traces a curve as it is held tight against the string (Figure 10.4.3b). The resulting curve will be an ellipse since the sum of the distances to the foci is a constant, namely, the total length of the string. Note that if the foci coincide, the ellipse reduces to a circle. For ellipses other than circles, the line segment through the foci and across the ellipse is called the major axis (Figure 10.4.3c), and the line segment across the ellipse, through the center, and perpendicular to the major axis is called the minor axis. The endpoints of the major axis are called vertices.

10.4.3 Definition A hyperbola is the set of all points in the plane, the difference of whose distances from two fixed distinct points is a given positive constant that is less than the distance between the fixed points.

The two fixed points are called the foci of the hyperbola, and the term “difference” that is used in the definition is understood to mean the distance to the farther focus minus the distance to the closer focus. As a result, the points on the hyperbola form two branches, each
“wrapping around” the closer focus (Figure 10.4.4a). The midpoint of the line segment joining the foci is called the center of the hyperbola, the line through the foci is called the focal axis, and the line through the center that is perpendicular to the focal axis is called the conjugate axis. The hyperbola intersects the focal axis at two points called the vertices.

Associated with every hyperbola is a pair of lines, called the asymptotes of the hyperbola. These lines intersect at the center of the hyperbola and have the property that as a point \( P \) moves along the hyperbola away from the center, the vertical distance between \( P \) and one of the asymptotes approaches zero (Figure 10.4.4b).

\[ \text{Figure 10.4.4} \]

**EQUATIONS OF PARABOLAS IN STANDARD POSITION**

It is traditional in the study of parabolas to denote the distance between the focus and the vertex by \( p \). The vertex is equidistant from the focus and the directrix, so the distance between the vertex and the directrix is also \( p \); consequently, the distance between the focus and the directrix is \( 2p \) (Figure 10.4.5). As illustrated in that figure, the parabola passes through two of the corners of a box that extends from the vertex to the focus along the axis of symmetry and extends \( 2p \) units above and \( 2p \) units below the axis of symmetry.

The equation of a parabola is simplest if the vertex is the origin and the axis of symmetry is along the \( x \)-axis or \( y \)-axis. The four possible such orientations are shown in Figure 10.4.6. These are called the standard positions of a parabola, and the resulting equations are called the standard equations of a parabola.

\[ \text{Figure 10.4.5} \]

\[ \text{Figure 10.4.6} \]
To illustrate how the equations in Figure 10.4.6 are obtained, we will derive the equation for the parabola with focus \((p, 0)\) and directrix \(x = -p\). Let \(P(x, y)\) be any point on the parabola. Since \(P\) is equidistant from the focus and directrix, the distances \(PF\) and \(PD\) in Figure 10.4.7 are equal; that is,

\[
PF = PD
\]

where \(D(-p, y)\) is the foot of the perpendicular from \(P\) to the directrix. From the distance formula, the distances \(PF\) and \(PD\) are

\[
PF = \sqrt{(x - p)^2 + y^2} \quad \text{and} \quad PD = \sqrt{(x + p)^2}
\]

Substituting in (1) and squaring yields

\[
(x - p)^2 + y^2 = (x + p)^2
\]

and after simplifying

\[
y^2 = 4px
\]

The derivations of the other equations in Figure 10.4.6 are similar.

### A TECHNIQUE FOR SKETCHING PARABOLAS

Parabolas can be sketched from their standard equations using four basic steps:

**Sketching a Parabola from Its Standard Equation**

**Step 1.** Determine whether the axis of symmetry is along the \(x\)-axis or the \(y\)-axis. Referring to Figure 10.4.6, the axis of symmetry is along the \(x\)-axis if the equation has a \(y^2\)-term, and it is along the \(y\)-axis if it has an \(x^2\)-term.

**Step 2.** Determine which way the parabola opens. If the axis of symmetry is along the \(x\)-axis, then the parabola opens to the right if the coefficient of \(x\) is positive, and it opens to the left if the coefficient is negative. If the axis of symmetry is along the \(y\)-axis, then the parabola opens up if the coefficient of \(y\) is positive, and it opens down if the coefficient is negative.

**Step 3.** Determine the value of \(p\) and draw a box extending \(p\) units from the origin along the axis of symmetry in the direction in which the parabola opens and extending \(2p\) units on each side of the axis of symmetry.

**Step 4.** Using the box as a guide, sketch the parabola so that its vertex is at the origin and it passes through the corners of the box (Figure 10.4.8).

#### Example 1

Sketch the graphs of the parabolas

\[
(a) \quad x^2 = 12y \quad \quad (b) \quad y^2 + 8x = 0
\]

and show the focus and directrix of each.

**Solution (a).** This equation involves \(x^2\), so the axis of symmetry is along the \(y\)-axis, and the coefficient of \(y\) is positive, so the parabola opens upward. From the coefficient of \(y\), we obtain \(4p = 12\) or \(p = 3\). Drawing a box extending \(p = 3\) units up from the origin and \(2p = 6\) units to the left and \(2p = 6\) units to the right of the \(y\)-axis, then using corners of the box as a guide, yields the graph in Figure 10.4.9.

The focus is \(p = 3\) units from the vertex along the axis of symmetry in the direction in which the parabola opens, so its coordinates are \((0, 3)\). The directrix is perpendicular to the axis of symmetry at a distance of \(p = 3\) units from the vertex on the opposite side from the focus, so its equation is \(y = -3\).
Solution (b). We first rewrite the equation in the standard form
\[ y^2 = -8x \]
This equation involves \( y^2 \), so the axis of symmetry is along the \( x \)-axis, and the coefficient of \( x \) is negative, so the parabola opens to the left. From the coefficient of \( x \) we obtain \( 4p = 8 \), so \( p = 2 \). Drawing a box extending \( p = 2 \) units left from the origin and \( 2p = 4 \) units above and \( 2p = 4 \) units below the \( x \)-axis, then using corners of the box as a guide, yields the graph in Figure 10.4.10.

Example 2 Find an equation of the parabola that is symmetric about the \( y \)-axis, has its vertex at the origin, and passes through the point \((5, 2)\).

Solution. Since the parabola is symmetric about the \( y \)-axis and has its vertex at the origin, the equation is of the form
\[ x^2 = 4py \]
where the sign depends on whether the parabola opens up or down. But the parabola must open up since it passes through the point \((5, 2)\), which lies in the first quadrant. Thus, the equation is of the form
\[ x^2 = 4py \] (5)
Since the parabola passes through \((5, 2)\), we must have \( 5^2 = 4p \cdot 2 \) or \( 4p = \frac{25}{2} \). Therefore, (5) becomes
\[ x^2 = \frac{25}{2}y \]

EQUATIONS OF ELLIPSES IN STANDARD POSITION
It is traditional in the study of ellipses to denote the length of the major axis by \( 2a \), the length of the minor axis by \( 2b \), and the distance between the foci by \( 2c \) (Figure 10.4.11). The number \( a \) is called the **semimajor axis** and the number \( b \) the **semiminor axis** (standard but odd terminology, since \( a \) and \( b \) are numbers, not geometric axes).

There is a basic relationship between the numbers \( a, b, \) and \( c \) that can be obtained by examining the sum of the distances to the foci from a point \( P \) at the end of the major axis and from a point \( Q \) at the end of the minor axis (Figure 10.4.12). From Definition 10.4.2, these sums must be equal, so we obtain
\[ 2\sqrt{b^2 + c^2} = (a - c) + (a + c) \]
from which it follows that
\[ a = \sqrt{b^2 + c^2} \] (6)
or, equivalently,
\[ c = \sqrt{a^2 - b^2} \] (7)
From (6), the distance from a focus to an end of the minor axis is \( a \) (Figure 10.4.13), which implies that for all points on the ellipse the sum of the distances to the foci is \( 2a \).

It also follows from (6) that \( a \geq b \) with the equality holding only when \( c = 0 \). Geometrically, this means that the major axis of an ellipse is at least as large as the minor axis and that the two axes have equal length only when the foci coincide, in which case the ellipse is a circle.

The equation of an ellipse is simplest if the center of the ellipse is at the origin and the foci are on the \( x \)-axis or \( y \)-axis. The two possible such orientations are shown in Figure 10.4.14.
These are called the **standard positions** of an ellipse, and the resulting equations are called the **standard equations** of an ellipse.

To illustrate how the equations in Figure 10.4.14 are obtained, we will derive the equation for the ellipse with foci on the $x$-axis. Let $P(x, y)$ be any point on that ellipse. Since the sum of the distances from $P$ to the foci is $2a$, it follows (Figure 10.4.15) that

$$PF' + PF = 2a$$

so

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

Transposing the second radical to the right side of the equation and squaring yields

$$(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$$

and, on simplifying,

$$\sqrt{(x - c)^2 + y^2} = a - \frac{c}{a}x$$  \hspace{1cm} (8)

Squaring again and simplifying yields

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

which, by virtue of (6), can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$  \hspace{1cm} (9)

Conversely, it can be shown that any point whose coordinates satisfy (9) has $2a$ as the sum of its distances from the foci, so that such a point is on the ellipse.
A TECHNIQUE FOR SKETCHING ELLIPSES

Ellipses can be sketched from their standard equations using three basic steps:

Sketching an Ellipse from Its Standard Equation

**Step 1.** Determine whether the major axis is on the x-axis or the y-axis. This can be ascertained from the sizes of the denominators in the equation. Referring to Figure 10.4.14, and keeping in mind that \( a^2 > b^2 \) (since \( a > b \)), the major axis is along the x-axis if \( x^2 \) has the larger denominator, and it is along the y-axis if \( y^2 \) has the larger denominator. If the denominators are equal, the ellipse is a circle.

**Step 2.** Determine the values of \( a \) and \( b \) and draw a box extending \( a \) units on each side of the center along the major axis and \( b \) units on each side of the center along the minor axis.

**Step 3.** Using the box as a guide, sketch the ellipse so that its center is at the origin and it touches the sides of the box where the sides intersect the coordinate axes (Figure 10.4.16).

> Example 3

Sketch the graphs of the ellipses

(a) \( \frac{x^2}{9} + \frac{y^2}{16} = 1 \)  
(b) \( x^2 + 2y^2 = 4 \)

showing the foci of each.

**Solution (a).** Since \( y^2 \) has the larger denominator, the major axis is along the y-axis. Moreover, since \( a^2 > b^2 \), we must have \( a^2 = 16 \) and \( b^2 = 9 \), so

\[
a = 4 \quad \text{and} \quad b = 3
\]

Drawing a box extending 4 units on each side of the origin along the y-axis and 3 units on each side of the origin along the x-axis as a guide yields the graph in Figure 10.4.17.

The foci lie \( c \) units on each side of the center along the major axis, where \( c \) is given by (7). From the values of \( a^2 \) and \( b^2 \) above, we obtain

\[
c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7} \approx 2.6
\]

Thus, the coordinates of the foci are \( (0, \sqrt{7}) \) and \( (0, -\sqrt{7}) \), since they lie on the y-axis.

**Solution (b).** We first rewrite the equation in the standard form

\[
\frac{x^2}{4} + \frac{y^2}{2} = 1
\]

Since \( x^2 \) has the larger denominator, the major axis lies along the x-axis, and we have \( a^2 = 4 \) and \( b^2 = 2 \). Drawing a box extending \( a = 2 \) units on each side of the origin along the x-axis and extending \( b = \sqrt{2} \approx 1.4 \) units on each side of the origin along the y-axis as a guide yields the graph in Figure 10.4.18.

From (7), we obtain

\[
c = \sqrt{a^2 - b^2} = \sqrt{2} \approx 1.4
\]

Thus, the coordinates of the foci are \( (\sqrt{2}, 0) \) and \( (-\sqrt{2}, 0) \), since they lie on the x-axis.
Example 4  Find an equation for the ellipse with foci \((0, \pm 2)\) and major axis with endpoints \((0, \pm 4)\).

**Solution.** From Figure 10.4.14, the equation has the form

\[
\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1
\]

and from the given information, \(a = 4\) and \(c = 2\). It follows from (6) that

\[
b^2 = a^2 - c^2 = 16 - 4 = 12
\]

so the equation of the ellipse is

\[
\frac{x^2}{12} + \frac{y^2}{16} = 1
\]

### EQUATIONS OF HYPERBOLAS IN STANDARD POSITION

It is traditional in the study of hyperbolas to denote the distance between the vertices by \(2a\), the distance between the foci by \(2c\) (Figure 10.4.19), and to define the quantity \(b\) as

\[
b = \sqrt{c^2 - a^2} \quad (10)
\]

This relationship, which can also be expressed as

\[
c = \sqrt{a^2 + b^2} \quad (11)
\]

is pictured geometrically in Figure 10.4.20. As illustrated in that figure, and as we will show later in this section, the asymptotes pass through the corners of a box extending \(b\) units on each side of the center along the conjugate axis and \(a\) units on each side of the center along the focal axis. The number \(a\) is called the **semifocal axis** of the hyperbola and the number \(b\) the **semiconjugate axis**. (As with the semimajor and semiminor axes of an ellipse, these are numbers, not geometric axes.)

If \(V\) is one vertex of a hyperbola, then, as illustrated in Figure 10.4.21, the distance from \(V\) to the farther focus minus the distance from \(V\) to the closer focus is

\[
[(c - a) + 2a] - (c - a) = 2a
\]

Thus, for all points on a hyperbola, the distance to the farther focus minus the distance to the closer focus is \(2a\).

The equation of a hyperbola has an especially convenient form if the center of the hyperbola is at the origin and the foci are on the \(x\)-axis or \(y\)-axis. The two possible such orientations are shown in Figure 10.4.22. These are called the **standard positions** of a hyperbola, and the resulting equations are called the **standard equations** of a hyperbola.

The derivations of these equations are similar to those already given for parabolas and ellipses, so we will leave them as exercises. However, to illustrate how the equations of the asymptotes are derived, we will derive those equations for the hyperbola

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

We can rewrite this equation as

\[
y^2 = \frac{b^2}{a^2}(x^2 - a^2)
\]

which is equivalent to the pair of equations

\[
y = \frac{b}{a}\sqrt{x^2 - a^2} \quad \text{and} \quad y = -\frac{b}{a}\sqrt{x^2 - a^2}
\]
Thus, in the first quadrant, the vertical distance between the line $y = (b/a)x$ and the hyperbola can be written as

$$\frac{b}{a}x = \frac{b}{a}\sqrt{x^2 - a^2}$$

(Figure 10.4.23). But this distance tends to zero as $x \to +\infty$ since

$$\lim_{x \to +\infty} \left( \frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2} \right) = \lim_{x \to +\infty} \frac{b}{a} \left( x - \sqrt{x^2 - a^2} \right)$$

$$= \lim_{x \to +\infty} \frac{b}{a} \left( x - \sqrt{x^2 - a^2} \right) \left( x + \sqrt{x^2 - a^2} \right)$$

$$= \lim_{x \to +\infty} \frac{ab}{x + \sqrt{x^2 - a^2}} = 0$$

The analysis in the remaining quadrants is similar.

A QUICK WAY TO FIND ASYMPTOTES

There is a trick that can be used to avoid memorizing the equations of the asymptotes of a hyperbola. They can be obtained, when needed, by replacing 1 by 0 on the right side of the hyperbola equation, and then solving for $y$ in terms of $x$. For example, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we would write

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad y^2 = \frac{b^2}{a^2}x^2 \quad \text{or} \quad y = \pm \frac{b}{a}x$$

which are the equations for the asymptotes.
A TECHNIQUE FOR SKETCHING HYPERBOLAS

Hyperbolas can be sketched from their standard equations using four basic steps:

**Sketching a Hyperbola from Its Standard Equation**

**Step 1.** Determine whether the focal axis is on the x-axis or the y-axis. This can be ascertained from the location of the minus sign in the equation. Referring to Figure 10.4.22, the focal axis is along the x-axis when the minus sign precedes the y²-term, and it is along the y-axis when the minus sign precedes the x²-term.

**Step 2.** Determine the values of a and b and draw a box extending a units on either side of the center along the focal axis and b units on either side of the center along the conjugate axis. (The squares of a and b can be read directly from the equation.)

**Step 3.** Draw the asymptotes along the diagonals of the box.

**Step 4.** Using the box and the asymptotes as a guide, sketch the graph of the hyperbola (Figure 10.4.24).

**Example 5** Sketch the graphs of the hyperbolas

(a) \( \frac{x^2}{4} - \frac{y^2}{9} = 1 \)  
(b) \( y^2 - x^2 = 1 \)

showing their vertices, foci, and asymptotes.

**Solution (a).** The minus sign precedes the y²-term, so the focal axis is along the x-axis. From the denominators in the equation we obtain

\[ a^2 = 4 \quad \text{and} \quad b^2 = 9 \]

Since a and b are positive, we must have \( a = 2 \) and \( b = 3 \). Recalling that the vertices lie a units on each side of the center on the focal axis, it follows that their coordinates in this case are \((2, 0)\) and \((-2, 0)\). Drawing a box extending a = 2 units along the x-axis on each side of the origin and \( b = 3 \) units on each side of the origin along the y-axis, then drawing the asymptotes along the diagonals of the box as a guide, yields the graph in Figure 10.4.25.

To obtain equations for the asymptotes, we replace 1 by 0 in the given equation; this yields

\[ \frac{x^2}{4} - \frac{y^2}{9} = 0 \quad \text{or} \quad y = \pm \frac{3}{2}x \]

The foci lie \( c \) units on each side of the center along the focal axis, where \( c \) is given by (11). From the values of \( a^2 \) and \( b^2 \) above we obtain

\[ c = \sqrt{a^2 + b^2} = \sqrt{4 + 9} = \sqrt{13} \approx 3.6 \]

Since the foci lie on the x-axis in this case, their coordinates are \((\sqrt{13}, 0)\) and \((-\sqrt{13}, 0)\).

**Solution (b).** The minus sign precedes the x²-term, so the focal axis is along the y-axis. From the denominators in the equation we obtain \( a^2 = 1 \) and \( b^2 = 1 \), from which it follows that

\[ a = 1 \quad \text{and} \quad b = 1 \]

Thus, the vertices are at \((0, -1)\) and \((0, 1)\). Drawing a box extending \( a = 1 \) unit on either side of the origin along the y-axis and \( b = 1 \) unit on either side of the origin along the x-axis, then drawing the asymptotes, yields the graph in Figure 10.4.26. Since the box is actually
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A hyperbola in which \(a = b\), as in part (b) of Example 5, is called an equilateral hyperbola. Such hyperbolas always have perpendicular asymptotes.

A square, the asymptotes are perpendicular and have equations \(y = \pm x\). This can also be seen by replacing 1 by 0 in the given equation, which yields \(y^2 - x^2 = 0\) or \(y = \pm x\). Also, 

\[c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2}\]

so the foci, which lie on the \(y\)-axis, are \((0, -\sqrt{2})\) and \((0, \sqrt{2})\).

\[\text{Example 6} \quad \text{Find the equation of the hyperbola with vertices (0, \(\pm 8\)) and asymptotes } y = \pm \frac{2}{3}x.\]

\[\text{Solution.} \quad \text{Since the vertices are on the } y\text{-axis, the equation of the hyperbola has the form } \left(\frac{y^2}{a^2}\right) - \left(\frac{x^2}{b^2}\right) = 1 \text{ and the asymptotes are } y = \pm \frac{a}{b}x.\]

From the locations of the vertices we have \(a = 8\), so the given equations of the asymptotes yield 

\[y = \pm \frac{8}{b}x = \pm \frac{4}{3}x\]

from which it follows that \(b = 6\). Thus, the hyperbola has the equation 

\[\frac{y^2}{64} - \frac{x^2}{36} = 1\]

\[\text{TRANSLATED CONICS}\]

Equations of conics that are translated from their standard positions can be obtained by replacing \(x\) by \(x - h\) and \(y\) by \(y - k\) in their standard equations. For a parabola, this translates the vertex from the origin to the point \((h, k)\); and for ellipses and hyperbolas, this translates the center from the origin to the point \((h, k)\).

\[\text{Parabolas with vertex } (h, k) \text{ and axis parallel to } x\text{-axis}\]

\[\begin{align*}
(y - k)^2 &= 4p(x - h) & \text{Opens right} \\
(y - k)^2 &= -4p(x - h) & \text{Opens left}
\end{align*}\]

\[\text{Parabolas with vertex } (h, k) \text{ and axis parallel to } y\text{-axis}\]

\[\begin{align*}
(x - h)^2 &= 4p(y - k) & \text{Opens up} \\
(x - h)^2 &= -4p(y - k) & \text{Opens down}
\end{align*}\]

\[\text{Ellipse with center } (h, k) \text{ and major axis parallel to } x\text{-axis}\]

\[\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad [b < a]\]

\[\text{Ellipse with center } (h, k) \text{ and major axis parallel to } y\text{-axis}\]

\[\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \quad [b < a]\]

\[\text{Hyperbola with center } (h, k) \text{ and focal axis parallel to } x\text{-axis}\]

\[\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1\]

\[\text{Hyperbola with center } (h, k) \text{ and focal axis parallel to } y\text{-axis}\]

\[\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1\]
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Example 7  Find an equation for the parabola that has its vertex at \((1, 2)\) and its focus at \((4, 2)\).

Solution.  Since the focus and vertex are on a horizontal line, and since the focus is to the right of the vertex, the parabola opens to the right and its equation has the form 
\[(y - k)^2 = 4p(x - h)\]

Since the vertex and focus are 3 units apart, we have \(p = 3\), and since the vertex is at \((h, k) = (1, 2)\), we obtain 
\[(y - 2)^2 = 12(x - 1)\]

Sometimes the equations of translated conics occur in expanded form, in which case we are faced with the problem of identifying the graph of a quadratic equation in \(x\) and \(y\):

\[Ax^2 + Cy^2 + Dx + Ey + F = 0\]  \hspace{1cm} (20)

The basic procedure for determining the nature of such a graph is to complete the squares of the quadratic terms and then try to match up the resulting equation with one of the forms of a translated conic.

Example 8  Describe the graph of the equation 
\[y^2 - 8x - 6y - 23 = 0\]

Solution.  The equation involves quadratic terms in \(y\) but none in \(x\), so we first take all of the \(y\)-terms to one side:
\[y^2 - 6y = 8x + 23\]

Next, we complete the square on the \(y\)-terms by adding 9 to both sides:
\[(y - 3)^2 = 8x + 32\]

Finally, we factor out the coefficient of the \(x\)-term to obtain
\[(y - 3)^2 = 8(x + 4)\]

This equation is of form (12) with \(h = -4, k = 3\), and \(p = 2\), so the graph is a parabola with vertex \((-4, 3)\) opening to the right. Since \(p = 2\), the focus is 2 units to the right of the vertex, which places it at the point \((-2, 3)\); and the directrix is 2 units to the left of the vertex, which means that its equation is \(x = -6\). The parabola is shown in Figure 10.4.27.

Example 9  Describe the graph of the equation 
\[16x^2 + 9y^2 - 64x - 54y + 1 = 0\]

Solution.  This equation involves quadratic terms in both \(x\) and \(y\), so we will group the \(x\)-terms and the \(y\)-terms on one side and put the constant on the other:
\[(16x^2 - 64x) + (9y^2 - 54y) = -1\]

Next, factor out the coefficients of \(x^2\) and \(y^2\) and complete the squares:
\[16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81\]

or
\[16(x - 2)^2 + 9(y - 3)^2 = 144\]
Finally, divide through by 144 to introduce a 1 on the right side:

$$\frac{(x - 2)^2}{9} + \frac{(y - 3)^2}{16} = 1$$

This is an equation of form (17), with $h = 2, k = 3, a^2 = 16$, and $b^2 = 9$. Thus, the graph of the equation is an ellipse with center $(2, 3)$ and major axis parallel to the $y$-axis. Since $a = 4$, the major axis extends 4 units above and 4 units below the center, so its endpoints are $(2, 7)$ and $(2, -1)$ (Figure 10.4.28). Since $b = 3$, the minor axis extends 3 units to the left and 3 units to the right of the center, so its endpoints are $(-1, 3)$ and $(5, 3)$. Since

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

the foci lie $\sqrt{7}$ units above and below the center, placing them at the points $(2, 3 + \sqrt{7})$ and $(2, 3 - \sqrt{7})$. ▲

**Example 10** Describe the graph of the equation

$$x^2 - y^2 - 4x + 8y - 21 = 0$$

**Solution.** This equation involves quadratic terms in both $x$ and $y$, so we will group the $x$-terms and the $y$-terms on one side and put the constant on the other:

$$(x^2 - 4x) - (y^2 - 8y) = 21$$

We leave it for you to verify by completing the squares that this equation can be written as

$$\frac{(x - 2)^2}{9} - \frac{(y - 4)^2}{9} = 1 \quad (21)$$

This is an equation of form (18) with $h = 2, k = 4, a^2 = 9$, and $b^2 = 9$. Thus, the equation represents a hyperbola with center $(2, 4)$ and focal axis parallel to the $x$-axis. Since $a = 3$, the vertices are located 3 units to the left and 3 units to the right of the center, or at the points $(-1, 4)$ and $(5, 4)$. From (11), $c = \sqrt{a^2 + b^2} = \sqrt{9 + 9} = 3\sqrt{2}$, so the foci are located $3\sqrt{2}$ units to the left and right of the center, or at the points $(2 - 3\sqrt{2}, 4)$ and $(2 + 3\sqrt{2}, 4)$.

The equations of the asymptotes may be found using the trick of replacing 1 by 0 in (21) to obtain

$$\frac{(x - 2)^2}{9} - \frac{(y - 4)^2}{9} = 0$$

This can be written as $y - 4 = \pm(x - 2)$, which yields the asymptotes

$$y = x + 2 \quad \text{and} \quad y = -x + 6$$

With the aid of a box extending $a = 3$ units left and right of the center and $b = 3$ units above and below the center, we obtain the sketch in Figure 10.4.29. ▲

**Reflection Properties of the Conic Sections**

Parabolas, ellipses, and hyperbolas have certain reflection properties that make them extremely valuable in various applications. In the exercises we will ask you to prove the following results.

**10.4.4 Theorem (Reflection Property of Parabolas)** The tangent line at a point $P$ on a parabola makes equal angles with the line through $P$ parallel to the axis of symmetry and the line through $P$ and the focus (Figure 10.4.30a).
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10.4.5 **Theorem (Reflection Property of Ellipses)** A line tangent to an ellipse at a point \( P \) makes equal angles with the lines joining \( P \) to the foci (Figure 10.4.30b).

10.4.6 **Theorem (Reflection Property of Hyperbolas)** A line tangent to a hyperbola at a point \( P \) makes equal angles with the lines joining \( P \) to the foci (Figure 10.4.30c).

**Applications of the Conic Sections**

Fermat’s principle in optics implies that light reflects off of a surface at an angle equal to its angle of incidence. (See Exercise 64 in Section 4.5.) In particular, if a reflecting surface is generated by revolving a parabola about its axis of symmetry, it follows from Theorem 10.4.4 that all light rays entering parallel to the axis will be reflected to the focus (Figure 10.4.31a); conversely, if a light source is located at the focus, then the reflected rays will all be parallel to the axis (Figure 10.4.31b). This principle is used in certain telescopes to reflect the approximately parallel rays of light from the stars and planets off of a parabolic mirror to an eyepiece at the focus; and the parabolic reflectors in flashlights and automobile headlights utilize this principle to form a parallel beam of light rays from a bulb placed at the focus. The same optical principles apply to radar signals and sound waves, which explains the parabolic shape of many antennas.

Visitors to various rooms in the United States Capitol Building and in St. Paul’s Cathedral in London are often astonished by the “whispering gallery” effect in which two people at opposite ends of the room can hear one another’s whispers very clearly. Such rooms have ceilings with elliptical cross sections and common foci. Thus, when the two people stand at the foci, their whispers are reflected directly to one another off of the elliptical ceiling.

**Hyperbolic navigation systems**, which were developed in World War II as navigational aids to ships, are based on the definition of a hyperbola. With these systems the ship receives
synchronized radio signals from two widely spaced transmitters with known positions. The ship’s electronic receiver measures the difference in reception times between the signals and then uses that difference to compute the difference \(2a\) between its distances from the two transmitters. This information places the ship somewhere on the hyperbola whose foci are at the transmitters and whose points have \(2a\) as the difference in their distances from the foci. By repeating the process with a second set of transmitters, the position of the ship can be approximated as the intersection of two hyperbolas (Figure 10.4.32). [The modern global positioning system (GPS) is based on the same principle.]

**Figure 10.4.32**

**QUICK CHECK EXERCISES 10.4** (See page 748 for answers.)

1. Identify the conic.
   (a) The set of points in the plane, the sum of whose distances to two fixed points is a positive constant greater than the distance between the fixed points is ________.
   (b) The set of points in the plane, the difference of whose distances to two fixed points is a positive constant less than the distance between the fixed points is ________.
   (c) The set of points in the plane that are equidistant from a fixed line and a fixed point not on the line is ________.

2. (a) The equation of the parabola with focus \((p, 0)\) and directrix \(x = -p\) is ________.
   (b) The equation of the parabola with focus \((0, p)\) and directrix \(x = -p\) is ________.

3. (a) Suppose that an ellipse has semimajor axis \(a\) and semimajor axis \(b\). Then for all points on the ellipse, the sum of the distances to the foci is equal to ________.
   (b) The two standard equations of an ellipse with semimajor axis \(a\) and semiminor axis \(b\) are ________ and ________.
   (c) Suppose that an ellipse has semimajor axis \(a\), semiminor axis \(b\), and foci \((\pm c, 0)\). Then \(c\) may be obtained from \(a\) and \(b\) by the equation \(c = \ldots\).

4. (a) Suppose that a hyperbola has semifocal axis \(a\) and semiconjugate axis \(b\). Then for all points on the hyperbola, the difference of the distance to the farther focus minus the distance to the closer focus is equal to ________.
   (b) The two standard equations of a hyperbola with semifocal axis \(a\) and semiconjugate axis \(b\) are ________ and ________.
   (c) Suppose that a hyperbola in standard position has semifocal axis \(a\), semiconjugate axis \(b\), and foci \((\pm c, 0)\). Then \(c\) may be obtained from \(a\) and \(b\) by the equation \(c = \ldots\). The equations of the asymptotes of this hyperbola are \(y = \pm \ldots\).
2. (a) Find the focus and directrix for each parabola in Exercise 1.
(b) Find the foci of the ellipses in Exercise 1.
(c) Find the foci and the equations of the asymptotes of the hyperbolas in Exercise 1.

3–6 Sketch the parabola, and label the focus, vertex, and directrix.
3. (a) \( y^2 = 4x \)  
(b) \( x^2 = -8y \)
4. (a) \( y^2 = -10x \)  
(b) \( x^2 = 4y \)
5. (a) \( (y - 1)^2 = -12(x + 4) \)  
(b) \( (x - 1)^2 = 2(y - \frac{1}{2}) \)
6. (a) \( y^2 - 6y - 2x + 1 = 0 \)  
(b) \( y = 4x^2 + 8x + 5 \)

7–10 Sketch the ellipse, and label the foci, vertices, and ends of the minor axis.
7. (a) \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \)  
(b) \( 9x^2 + y^2 = 9 \)
8. (a) \( \frac{x^2}{25} + \frac{y^2}{4} = 1 \)  
(b) \( 4x^2 + y^2 = 36 \)
9. (a) \( (x + 3)^2 + 4(y - 5)^2 = 16 \)  
(b) \( \frac{1}{4}x^2 + \frac{1}{2}(y + 2)^2 - 1 = 0 \)
10. (a) \( 9x^2 + 4y^2 - 18x + 24y + 9 = 0 \)  
(b) \( 5x^2 + 9y^2 + 20x - 54y = -56 \)

11–14 Sketch the hyperbola, and label the vertices, foci, and asymptotes.
11. (a) \( \frac{x^2}{16} - \frac{y^2}{9} = 1 \)  
(b) \( 9y^2 - x^2 = 36 \)
12. (a) \( \frac{x^2}{9} - \frac{y^2}{25} = 1 \)  
(b) \( 16x^2 - 25y^2 = 400 \)
13. (a) \( \frac{(y + 4)^2}{3} - \frac{(x - 2)^2}{5} = 1 \)  
(b) \( 16(x + 1)^2 - 8(y - 3)^2 = 16 \)
14. (a) \( x^2 - 4y^2 + 2x + 8y - 7 = 0 \)  
(b) \( 16x^2 - y^2 - 32x - 6y = 57 \)

15–18 Find an equation for the parabola that satisfies the given conditions.
15. (a) Vertex (0, 0); focus (3, 0).
(b) Vertex (0, 0); directrix \( y = \frac{1}{2} \).
16. (a) Focus (6, 0); directrix \( x = -6 \).
(b) Focus (1, 1); directrix \( y = -2 \).
17. Axis \( y = 0 \); passes through (3, 2) and (2, \(-\sqrt{2}\)).

18. Vertex (5, -3); axis parallel to the \( y \)-axis; passes through (9, 5).

19–22 Find an equation for the ellipse that satisfies the given conditions. ■
19. (a) Ends of major axis \((\pm 3, 0)\); ends of minor axis \((0, \pm 2)\).
(b) Length of minor axis 8; foci \((0, \pm 3)\).
20. (a) Foci \((\pm 1, 0); \ b = \sqrt{2} \).
(b) \( c = 2\sqrt{3}; \ a = 4; \) center at the origin; foci on a coordinate axis (two answers).
21. (a) Ends of major axis \((0, \pm 6)\); passes through \((-3, 2)\).
(b) Foci \((-1, 1)\) and \((-1, 3)\); minor axis of length 4.
22. (a) Center at \((0, 0); \) major and minor axes along the coordinate axes; passes through \((3, 2)\) and \((1, 6)\).
(b) Foci \((2, 1)\) and \((2, -3)\); major axis of length 6.

23–26 Find an equation for a hyperbola that satisfies the given conditions. [Note: In some cases there may be more than one hyperbola.]
23. (a) Vertices \((\pm 2, 0); \) foci \((\pm 3, 0)\).
(b) Vertices \((0, \pm 2); \) asymptotes \(y = \pm \sqrt{2}x\).
24. (a) Asymptotes \(y = \pm \frac{3}{2}x; \ b = 4 \).
(b) Foci \((0, \pm 5); \) asymptotes \(y = \pm 2x\).
25. (a) Asymptotes \(y = \pm \frac{3}{2}x; \ c = 5 \).
(b) Foci \((\pm 3, 0); \) asymptotes \(y = \pm 2x\).
26. (a) Vertices \((0, 6)\) and \((6, 6); \) foci 10 units apart.
(b) Asymptotes \(y = x - 2\) and \(y = -x + 4\); passes through the origin.

27–30 True–False Determine whether the statement is true or false. Explain your answer. ■
27. A hyperbola is the set of all points in the plane that are equidistant from a fixed line and a fixed point not on the line.
28. If an ellipse is not a circle, then the foci of an ellipse lie on the major axis of the ellipse.
29. If a parabola has equation \( y^2 = 4px \), where \( p \) is a positive constant, then the perpendicular distance from the parabola’s focus to its directrix is \( p \).
30. The hyperbola \( (y^2/a^2) - x^2 = 1 \) has asymptotes the lines \( y = \pm x/a \).
31. (a) As illustrated in the accompanying figure, a parabolic arch spans a road 40 ft wide. How high is the arch if a center section of the road 20 ft wide has a minimum clearance of 12 ft?
(b) How high would the center be if the arch were the upper half of an ellipse?
32. (a) Find an equation for the parabolic arch with base $b$ and height $h$, shown in the accompanying figure.
(b) Find the area under the arch.

![Figure Ex-32]

33. Show that the vertex is the closest point on a parabola to the focus. [Suggestion: Introduce a convenient coordinate system and use Definition 10.4.1.]

34. As illustrated in the accompanying figure, suppose that a comet moves in a parabolic orbit with the Sun at its focus and that the line from the Sun to the comet makes an angle of $60^\circ$ with the axis of the parabola when the comet is 40 million miles from the center of the Sun. Use the result in Exercise 33 to determine how close the comet will come to the center of the Sun.

35. For the parabolic reflector in the accompanying figure, how far from the vertex should the light source be placed to produce a beam of parallel rays?

![Figure Ex-34]

36. (a) Show that the right and left branches of the hyperbola
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]
can be represented parametrically as
\[ x = a \cosh t, \quad y = b \sinh t \quad (-\infty < t < +\infty) \]
\[ x = -a \cosh t, \quad y = b \sinh t \quad (-\infty < t < +\infty) \]
(b) Use a graphing utility to generate both branches of the hyperbola $x^2 - y^2 = 1$ on the same screen.

![Figure Ex-35]

37. (a) Show that the right and left branches of the hyperbola
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]
can be represented parametrically as
\[ x = a \sec t, \quad y = b \tan t \quad (-\pi/2 < t < \pi/2) \]
\[ x = -a \sec t, \quad y = b \tan t \quad (-\pi/2 < t < \pi/2) \]
(b) Use a graphing utility to generate both branches of the hyperbola $x^2 - y^2 = 1$ on the same screen.

![Figure Ex-44]

38. Find an equation of the parabola traced by a point that moves so that its distance from $(2, 4)$ is the same as its distance to the $x$-axis.

39. Find an equation of the ellipse traced by a point that moves so that the sum of its distances to $(4, 1)$ and $(4, 5)$ is 12.

40. Find the equation of the hyperbola traced by a point that moves so that the difference between its distances to $(0, 0)$ and $(1, 1)$ is 1.

41. Suppose that the base of a solid is elliptical with a major axis of length 9 and a minor axis of length 4. Find the volume of the solid if the cross sections perpendicular to the major axis are squares (see the accompanying figure).

42. Suppose that the base of a solid is elliptical with a major axis of length 9 and a minor axis of length 4. Find the volume of the solid if the cross sections perpendicular to the minor axis are equilateral triangles (see the accompanying figure).

43. Show that an ellipse with semimajor axis $a$ and semiminor axis $b$ has area $A = \pi ab$.

44. Show that if a plane is not parallel to the axis of a right circular cylinder, then the intersection of the plane and cylinder is an ellipse (possibly a circle). [Hint: Let $\theta$ be the angle shown in the accompanying figure, introduce coordinate axes as shown, and express $x'$ and $y'$ in terms of $x$ and $y$.]

45. As illustrated in the accompanying figure on the next page, a carpenter needs to cut an elliptical hole in a sloped roof through which a circular vent pipe of diameter $D$ is to be inserted vertically. The carpenter wants to draw the outline of the hole on the roof using a pencil, two tacks, and a piece of string (as in Figure 10.4.3). The center point of the ellipse is known, and common sense suggests that its major axis must be perpendicular to the drip line of the roof. The carpenter needs to determine the length $L$ of the string and the distance $T$ between a tack and the center point. The architect’s plans show that the pitch of the roof is $p$ (pitch = rise over run; see the accompanying figure). Find $T$ and $L$ in terms of $D$ and $p$.

Source: This exercise is based on an article by William H. Enos, which appeared in the Mathematics Teacher, Feb. 1991, p. 148.
46. As illustrated in the accompanying figure, suppose that two observers are stationed at the points \(F_1(c, 0)\) and \(F_2(−c, 0)\) in an xy-coordinate system. Suppose also that the sound of an explosion in the xy-plane is heard by the \(F_1\) observer \(t\) seconds before it is heard by the \(F_2\) observer. Assuming that the speed of sound is a constant \(v\), show that the explosion occurred somewhere on the hyperbola

\[
\frac{x^2}{v^2r^2/4} - \frac{y^2}{c^2 - (v^2r^2/4)} = 1
\]

As illustrated in the accompanying figure, suppose that the region \(R\) is to have a height of 49 ft. Let

\[
x = \sqrt{a^2 + b^2}
\]

(a) Find the volume of the tower.
(b) Find the lateral surface area of the tower.

47. As illustrated in the accompanying figure, suppose that two transmitting stations are positioned 100 km apart at points \(F_1(50, 0)\) and \(F_2(−50, 0)\) on a straight shoreline in an xy-coordinate system. Suppose also that a ship is traveling parallel to the shoreline but 200 km at sea. Find the coordinates of the ship if the stations transmit a pulse simultaneously, but the pulse from station \(F_1\) is received by the ship 100 microseconds sooner than the pulse from station \(F_2\). [Assume that the pulses travel at the speed of light (299,792,458 m/s).]

48. A nuclear cooling tower is to have a height of \(h\) feet and the shape of the solid that is generated by revolving the region \(R\) enclosed by the right branch of the hyperbola \(1521x^2 - 225y^2 = 342,225\) and the lines \(x = 0\), \(y = −h/2\), and \(y = h/2\) about the \(y\)-axis.
(a) Find the volume of the tower.
(b) Find the lateral surface area of the tower.

49. Let \(R\) be the region that is above the \(x\)-axis and enclosed between the curve \(b^2x^2 − a^2y^2 = a^2b^2\) and the line \(x = \sqrt{a^2 + b^2}\).

10.4 Conic Sections

(a) Sketch the solid generated by revolving \(R\) about the \(x\)-axis, and find its volume.
(b) Sketch the solid generated by revolving \(R\) about the \(y\)-axis, and find its volume.

50. As illustrated in the accompanying figure, the tank of an oil truck is 18 ft long and has elliptical cross sections that are 6 ft wide and 4 ft high.
(a) Show that the volume \(V\) of oil in the tank (in cubic feet) when it is filled to a depth of \(h\) feet is
\[
V = 27 \left[ 4 \sin^{-1} \frac{h - 2}{2} + \frac{h - 2}{\sqrt{h^2 - (h - 2)^2}} + \frac{2h - 4}{\sqrt{h^2 - (h - 2)^2}} \right]
\]
(b) Use the numerical root-finding capability of a CAS to determine how many inches from the bottom of a dipstick the calibration marks should be placed to indicate when the tank is \(\frac{1}{4}\), \(\frac{1}{2}\), and \(\frac{3}{4}\) full.
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where $A$ and $C$ are not both 0. Show by completing the square:
(a) If $AC > 0$, then the equation represents an ellipse, a circle, a point, or has no graph.
(b) If $AC < 0$, then the equation represents a hyperbola or a pair of intersecting lines.
(c) If $AC = 0$, then the equation represents a parabola, a pair of parallel lines, or has no graph.

59. In each part, use the result in Exercise 58 to make a statement about the graph of the equation, and then check your conclusion by completing the square and identifying the graph.
(a) $x^2 - 5y^2 - 2x - 10y - 9 = 0$
(b) $x^2 - 3y^2 - 6y - 3 = 0$
(c) $4x^2 + 8y^2 + 16x + 16y + 20 = 0$
(d) $3x^2 + y^2 + 12x + 2y + 13 = 0$
(e) $x^2 + 8x + 2y + 14 = 0$
(f) $5x^2 + 40x + 2y + 94 = 0$

60. Derive the equation $x^2 = 4py$ in Figure 10.4.6.

61. Derive the equation $(x^2/b^2) + (y^2/a^2) = 1$ given in Figure 10.4.14.

62. Derive the equation $(x^2/a^2) - (y^2/b^2) = 1$ given in Figure 10.4.22.

63. Prove Theorem 10.4.4. [Hint: Choose coordinate axes so that the parabola has the equation $x^2 = 4py$. Show that the tangent line at $P(x_0, y_0)$ intersects the $y$-axis at $Q(0, -y_0)$ and that the triangle whose three vertices are at $P, Q,$ and the focus is isosceles.]

64. Given two intersecting lines, let $L_2$ be the line with the larger angle of inclination $\phi_2$, and let $L_1$ be the line with the smaller angle of inclination $\phi_1$. We define the angle $\theta$ between $L_1$ and $L_2$ by $\theta = \phi_2 - \phi_1$. (See the accompanying figure.)
(a) Prove: If $L_1$ and $L_2$ are not perpendicular, then

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1m_2}$$

where $L_1$ and $L_2$ have slopes $m_1$ and $m_2$.
(b) Prove Theorem 10.4.5. [Hint: Introduce coordinates so that the equation $(x^2/a^2) + (y^2/b^2) = 1$ describes the ellipse, and use part (a)].
(c) Prove Theorem 10.4.6. [Hint: Introduce coordinates so that the equation $(x^2/a^2) - (y^2/b^2) = 1$ describes the hyperbola, and use part (a)].

65. Writing Suppose that you want to draw an ellipse that has given values for the lengths of the major and minor axes by using the method shown in Figure 10.4.3b. Assuming that the axes are drawn, explain how a compass can be used to locate the positions for the tacks.

66. Writing List the forms for standard equations of parabolas, ellipses, and hyperbolas, and write a summary of techniques for sketching conic sections from their standard equations.

Quick Check Answers 10.4

1. (a) an ellipse (b) a hyperbola (c) a parabola
2. (a) $y^2 = 4px$ (b) $x^2 = 4py$
3. (a) $2a/b$ (b) $x^2/a^2 + y^2/b^2 = 1$; $x^2/b^2 + y^2/a^2 = 1$ (c) $\sqrt{a^2 - b^2}$
4. (a) $2a/b$ (b) $x^2/a^2 - y^2/b^2 = 1$; $y^2/b^2 - x^2/a^2 = 1$ (c) $\sqrt{a^2 + b^2}$; $b/a$

10.5 Rotation of Axes; Second-Degree Equations

In the preceding section we obtained equations of conic sections with axes parallel to the coordinate axes. In this section we will study the equations of conics that are “tilted” relative to the coordinate axes. This will lead us to investigate rotations of coordinate axes.

Quadratic Equations in $x$ and $y$

We saw in Examples 8 to 10 of the preceding section that equations of the form

$$Ax^2 + Cxy + Dy + F = 0$$

(1)

can represent conic sections. Equation (1) is a special case of the more general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

(2)

which, if $A$, $B$, and $C$ are not all zero, is called a quadratic equation in $x$ and $y$. It is usually the case that the graph of any second-degree equation is a conic section. If $B = 0,$...
then (2) reduces to (1) and the conic section has its axis or axes parallel to the coordinate axes. However, if $B \neq 0$, then (2) contains a cross-product term $Bxy$, and the graph of the conic section represented by the equation has its axis or axes “tilted” relative to the coordinate axes. As an illustration, consider the ellipse with foci $F_1(1, 2)$ and $F_2(-1, -2)$ and such that the sum of the distances from each point $P(x, y)$ on the ellipse to the foci is 6 units. Expressing this condition as an equation, we obtain (Figure 10.5.1)

\[ \sqrt{(x - 1)^2 + (y - 2)^2} + \sqrt{(x + 1)^2 + (y + 2)^2} = 6 \]

Squaring both sides, then isolating the remaining radical, then squaring again ultimately yields

\[ 8x^2 - 4xy + 5y^2 = 36 \]

as the equation of the ellipse. This is of form (2) with $A = 8$, $B = -4$, $C = 5$, $D = 0$, $E = 0$, and $F = -36$.

### ROTATION OF AXES

To study conics that are tilted relative to the coordinate axes it is frequently helpful to rotate the coordinate axes, so that the rotated coordinate axes are parallel to the axes of the conic. Before we can discuss the details, we need to develop some ideas about rotation of coordinate axes.

In Figure 10.5.2a the axes of an $xy$-coordinate system have been rotated about the origin through an angle $\theta$ to produce a new $x'y'$-coordinate system. As shown in the figure, each point $P$ in the plane has coordinates $(x', y')$ as well as coordinates $(x, y)$. To see how the two are related, let $r$ be the distance from the common origin to the point $P$, and let $\alpha$ be the angle shown in Figure 10.5.2b. It follows that

\[ x = r \cos(\theta + \alpha), \quad y = r \sin(\theta + \alpha) \quad (3) \]

and

\[ x' = r \cos \alpha, \quad y' = r \sin \alpha \quad (4) \]

Using familiar trigonometric identities, the relationships in (3) can be written as

\[ x = r \cos \theta \cos \alpha - r \sin \theta \sin \alpha \]
\[ y = r \sin \theta \cos \alpha + r \cos \theta \sin \alpha \]

and on substituting (4) in these equations we obtain the following relationships called the rotation equations:

\[ x = x' \cos \theta - y' \sin \theta \]
\[ y = x' \sin \theta + y' \cos \theta \quad (5) \]

\[ \textbf{Example 1} \quad \text{Suppose that the axes of an } xy \text{-coordinate system are rotated through an angle of } \theta = 45^\circ \text{ to obtain an } x'y' \text{-coordinate system. Find the equation of the curve } x^2 - xy + y^2 - 6 = 0 \text{ in } x'y' \text{-coordinates.} \]
We will always choose an angle \( \theta \) in the interval \( 0 < \theta < \pi/2 \).

\[ \cos \theta = \frac{\sqrt{2}}{2}, \quad \sin \theta = \frac{\sqrt{2}}{2} \]

Substituting these into the given equation yields

\[ \frac{x^2 - 2x'y' + y'^2}{2} + \frac{x^2 + y^2}{2} = 6 \]

which is the equation of an ellipse (Figure 10.5.3).

Example 2 Find the new coordinates of the point \((2, 4)\) if the coordinate axes are rotated through an angle of \( \theta = 30^\circ \).

Solution. Using the rotation equations in (6) with \( x = 2, y = 4, \cos \theta = \cos 30^\circ = \sqrt{3}/2, \) and \( \sin \theta = \sin 30^\circ = 1/2, \) we obtain

\[ x' = 2(\sqrt{3}/2) + 4(1/2) = \sqrt{3} + 2 \]
\[ y' = -2(1/2) + 4(\sqrt{3}/2) = -1 + 2\sqrt{3} \]

Thus, the new coordinates are \((\sqrt{3} + 2, -1 + 2\sqrt{3})\).

ELIMINATING THE CROSS-PRODUCT TERM

In Example 1 we were able to identify the curve \( x^2 - xy + y^2 - 6 = 0 \) as an ellipse because the rotation of axes eliminated the xy-term, thereby reducing the equation to a familiar form. This occurred because the new \( x'y' \)-axes were aligned with the axes of the ellipse. The following theorem tells how to determine an appropriate rotation of axes to eliminate the cross-product term of a second-degree equation in \( x \) and \( y \).

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \] (7)

is such that \( B \neq 0, \) and if an \( x'y' \)-coordinate system is obtained by rotating the \( xy \)-axes through an angle \( \theta \) satisfying

\[ \cot 2\theta = \frac{A - C}{B} \] (8)

then, in \( x'y' \)-coordinates, Equation (7) will have the form

\[ A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \]

Proof Substituting (5) into (7) and simplifying yields

\[ A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \]
where
\[ A' = A \cos^2 \theta + 2B \cos \theta \sin \theta + C \sin^2 \theta \]
\[ B' = B(\cos^2 \theta - \sin^2 \theta) + 2(C - A) \sin \theta \cos \theta \]
\[ C' = A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta \]
\[ D' = D \cos \theta + E \sin \theta \]
\[ E' = -D \sin \theta + E \cos \theta \]
\[ F' = F \]

(Verify.) To complete the proof we must show that \( B' = 0 \) if \( \cot 2\theta = \frac{A - C}{B} \)
or, equivalently,
\[ \frac{\cos 2\theta}{\sin 2\theta} = \frac{A - C}{B} \]  

However, by using the trigonometric double-angle formulas, we can rewrite \( B' \) in the form
\[ B' = B \cos 2\theta - (A - C) \sin 2\theta \]
Thus, \( B' = 0 \) if \( \theta \) satisfies (10).

\[ \blacktriangleleft \text{Example 3} \quad \text{Identify and sketch the curve } xy = 1. \]

\[ \textbf{Solution.} \quad \text{As a first step, we will rotate the coordinate axes to eliminate the cross-product term. Comparing the given equation to (7), we have} \]
\[ A = 0, \quad B = 1, \quad C = 0 \]
Thus, the desired angle of rotation must satisfy
\[ \cot 2\theta = \frac{A - C}{B} = \frac{0 - 0}{1} = 0 \]
This condition can be met by taking \( 2\theta = \pi/2 \) or \( \theta = \pi/4 = 45^\circ \). Making the substitutions \( \cos \theta = \cos 45^\circ = 1/\sqrt{2} \) and \( \sin \theta = \sin 45^\circ = 1/\sqrt{2} \) in (5) yields
\[ x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}} \]
Substituting these in the equation \( xy = 1 \) yields
\[ \left( \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \right) \left( \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}} \right) = 1 \quad \text{or} \quad \frac{x'^2}{2} - \frac{y'^2}{2} = 1 \]
which is the equation in the \( x'y' \)-coordinate system of an equilateral hyperbola with vertices at \((\sqrt{2}, 0)\) and \((-\sqrt{2}, 0)\) in that coordinate system (Figure 10.5.4). \( \blacktriangleleft \)

In problems where it is inconvenient to solve
\[ \cot 2\theta = \frac{A - C}{B} \]
for \( \theta \), the values of \( \sin \theta \) and \( \cos \theta \) needed for the rotation equations can be obtained by first calculating \( \cos 2\theta \) and then computing \( \sin \theta \) and \( \cos \theta \) from the identities
\[ \sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} \quad \text{and} \quad \cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}} \]
2. If the equation
\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]
is such that \( B \neq 0 \), then the \( xy \)-term in this equation can be eliminated by a rotation of axes through an angle \( \theta \) satisfying \( \cot 2\theta = \ldots \).

3. In each part, determine a rotation angle \( \theta \) that will eliminate the \( xy \)-term.
   (a) \( 2x^2 + xy + 2y^2 + x - y = 0 \)
   (b) \( x^2 + 2\sqrt{3}xy + 3y^2 = 2x + y = 1 \)
   (c) \( 3x^2 + \sqrt{3}xy + 2y^2 + y = 0 \)

4. Express \( 2x^2 + xy + 2y^2 = 1 \) in the \( x'y' \)-coordinate system obtained by rotating the \( xy \)-coordinate system through the angle \( \theta = \pi/4 \).
10.5 Rotation of Axes; Second-Degree Equations

EXERCISE SET 10.5

1. Let an $x'y'$-coordinate system be obtained by rotating an $xy$-coordinate system through an angle of $\theta = 60^\circ$.
   (a) Find the $x'y'$-coordinates of the point whose $xy$-coordinates are $(-2, 6)$.
   (b) Find an equation of the curve $\sqrt{3}xy + y^2 = 6$ in $x'y'$-coordinates.
   (c) Sketch the curve in part (b), showing both $xy$-axes and $x'y'$-axes.

2. Let an $x'y'$-coordinate system be obtained by rotating an $xy$-coordinate system through an angle of $\theta = 30^\circ$.
   (a) Find the $x'y'$-coordinates of the point whose $xy$-coordinates are $(1, -\sqrt{3})$.
   (b) Find an equation of the curve $2x^2 + 2\sqrt{3}xy = 3$ in $x'y'$-coordinates.
   (c) Sketch the curve in part (b), showing both $xy$-axes and $x'y'$-axes.

3–12 Rotate the coordinate axes to remove the $xy$-term. Then identify the type of conic and sketch its graph.
   3. $xy = -9$
   4. $x^2 - xy + y^2 - 2 = 0$
   5. $x^2 + 4xy - 2y^2 - 6 = 0$
   6. $31x^2 + 10\sqrt{3}xy + 21y^2 - 144 = 0$
   7. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$
   8. $34x^2 - 24xy + 41y^2 - 25 = 0$
   9. $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$
   10. $5x^2 - 6xy + 5y^2 - 8\sqrt{3}x + 8\sqrt{3}y = 8$
   11. $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$
   12. $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$

13. Let an $x'y'$-coordinate system be obtained by rotating an $xy$-coordinate system through an angle of $45^\circ$. Use (6) to find an equation of the curve $3x^2 + y^2 = 6$ in $xy$-coordinates.

14. Let an $x'y'$-coordinate system be obtained by rotating an $xy$-coordinate system through an angle of $30^\circ$. Use (5) to find an equation in $x'y'$-coordinates of the curve $y = x^2$.

15. Let an $x'y'$-coordinate system be obtained by rotating an $xy$-coordinate system through an angle $\theta$. Prove: For every value of $\theta$, the equation $x^2 + y^2 = r^2$ becomes the equation $x'^2 + y'^2 = r^2$. Give a geometric explanation.

16. Derive (6) by solving the rotation equations in (5) for $x'$ and $y'$ in terms of $x$ and $y$.

17. Let an $x'y'$-coordinate system be obtained by rotating an $xy$-coordinate system through an angle $\theta$. Explain how to find the $xy$-coordinates of a point whose $x'y'$-coordinates are known.

18. Let an $x'y'$-coordinate system be obtained by rotating an $xy$-coordinate system through an angle $\theta$. Explain how to find the $xy$-equation of a line whose $x'y'$-equation is known.

19–22 Show that the graph of the given equation is a parabola. Find its vertex, focus, and directrix.

19. $x^2 + 2xy + y^2 + 4\sqrt{2}x - 4\sqrt{2}y = 0$
20. $x^2 - 2\sqrt{3}xy + 3y^2 - 8\sqrt{3}x - 8y = 0$
21. $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$
22. $x^2 + 2\sqrt{3}xy + 3y^2 + 16\sqrt{3}x - 16y - 96 = 0$

23–26 Show that the graph of the given equation is an ellipse. Find its foci, vertices, and the ends of its minor axis.

23. $288x^2 - 168xy + 337y^2 - 3600 = 0$
24. $25x^2 - 14xy + 25y^2 - 288 = 0$
25. $31x^2 + 10\sqrt{3}xy + 21y^2 - 32x + 32\sqrt{3}y - 80 = 0$
26. $43x^2 - 14\sqrt{3}xy + 57y^2 - 36\sqrt{3}x - 36y - 540 = 0$

27–30 Show that the graph of the given equation is a hyperbola. Find its foci, vertices, and asymptotes.

27. $x^2 - 10\sqrt{3}xy + 11y^2 + 64 = 0$
28. $17x^2 - 312xy + 108y^2 - 900 = 0$
29. $32x^2 - 52xy - 7x^2 + 72\sqrt{3}x - 144\sqrt{3}y + 900 = 0$
30. $2\sqrt{3}y^2 + 5\sqrt{2}x^2 + 2\sqrt{3}x^2 + 18x + 18y + 36\sqrt{2} = 0$

31. Show that the graph of the equation
   $$\sqrt{x} + \sqrt{y} = 1$$
   is a portion of a parabola. [Hint: First rationalize the equation and then perform a rotation of axes.]

FOCUS ON CONCEPTS

32. Derive the expression for $B'$ in (9).
33. Use (9) to prove that $B^2 - 4AC = B'^2 - 4A'C'$ for all values of $\theta$.
34. Use (9) to prove that $A + C = A' + C'$ for all values of $\theta$.
35. Prove: If $A = C$ in (7), then the cross-product term can be eliminated by rotating through $45^\circ$.
36. Prove: If $B \neq 0$, then the graph of $x^2 + Bxy + F = 0$ is a hyperbola if $F \neq 0$ and two intersecting lines if $F = 0$. 
QUICK CHECK ANSWERS 10.5

1. (a) \(x' \cos \theta - y' \sin \theta\); \(x' \sin \theta + y' \cos \theta\) (b) \(x \cos \theta + y \sin \theta\); \(-x \sin \theta + y \cos \theta\)
2. \(A - C \over B\)
3. (a) \(\pi \over 4\) (b) \(\pi \over 3\) (c) \(\pi \over 6\)

4. \(5x^2 + 3y^2 = 2\)

10.6 CONIC SECTIONS IN POLAR COORDINATES

It will be shown later in the text that if an object moves in a gravitational field that is directed toward a fixed point (such as the center of the Sun), then the path of that object must be a conic section with the fixed point at a focus. For example, planets in our solar system move along elliptical paths with the Sun at a focus, and the comets move along parabolic, elliptical, or hyperbolic paths with the Sun at a focus, depending on the conditions under which they were born. For applications of this type it is usually desirable to express the equations of the conic sections in polar coordinates with the pole at a focus. In this section we will show how to do this.

THE FOCUS–DIRECTRIX CHARACTERIZATION OF CONICS

To obtain polar equations for the conic sections we will need the following theorem.

10.6.1 Theorem (Focus–Directrix Property of Conics) Suppose that a point \(P\) moves in the plane determined by a fixed point (called the focus) and a fixed line (called the directrix), where the focus does not lie on the directrix. If the point moves in such a way that its distance to the focus divided by its distance to the directrix is some constant \(e\) (called the eccentricity), then the curve traced by the point is a conic section. Moreover, the conic is

(a) a parabola if \(e = 1\)  
(b) an ellipse if \(0 < e < 1\)  
(c) a hyperbola if \(e > 1\).

We will not give a formal proof of this theorem; rather, we will use the specific cases in Figure 10.6.1 to illustrate the basic ideas. For the parabola, we will take the directrix to be \(x = -p\), as usual; and for the ellipse and the hyperbola we will take the directrix to be \(x = a^2/c\). We want to show in all three cases that if \(P\) is a point on the graph, \(F\) is the focus, and \(D\) is the directrix, then the ratio \(PF/PD\) is some constant \(e\), where \(e = 1\) for the parabola, \(0 < e < 1\) for the ellipse, and \(e > 1\) for the hyperbola. We will give the arguments for the parabola and ellipse and leave the argument for the hyperbola as an exercise.

\[x = -p\]
\[x = a^2/c\]
For the parabola, the distance $PF$ to the focus is equal to the distance $PD$ to the directrix, so that $PF/PD = 1$, which is what we wanted to show. For the ellipse, we rewrite Equation (8) of Section 10.4 as

$$\sqrt{(x - c)^2 + y^2} = a - \frac{c}{a}x = \frac{c}{a}\left(\frac{a^2}{c} - x\right)$$

But the expression on the left side is the distance $PF$, and the expression in the parentheses on the right side is the distance $PD$, so we have shown that

$$PF = \frac{c}{a}PD$$

Thus, $PF/PD$ is constant, and the eccentricity is

$$e = \frac{c}{a}$$

If we rule out the degenerate case where $a = 0$ or $c = 0$, then it follows from Formula (7) of Section 10.4 that $0 < c < a$, so $0 < e < 1$, which is what we wanted to show.

We will leave it as an exercise to show that the eccentricity of the hyperbola in Figure 10.6.1 is also given by Formula (1), but in this case it follows from Formula (11) of Section 10.4 that $c > a$, so $e > 1$.

**ECCENTRICITY OF AN ELLIPSE AS A MEASURE OF FLATNESS**

The eccentricity of an ellipse can be viewed as a measure of its flatness—as $e$ approaches 0 the ellipses become more and more circular, and as $e$ approaches 1 they become more and more flat (Figure 10.6.2). Table 10.6.1 shows the orbital eccentricities of various celestial objects. Note that most of the planets actually have fairly circular orbits.

<table>
<thead>
<tr>
<th>Table 10.6.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>CELESTIAL BODY</td>
</tr>
<tr>
<td>Mercury</td>
</tr>
<tr>
<td>Venus</td>
</tr>
<tr>
<td>Earth</td>
</tr>
<tr>
<td>Mars</td>
</tr>
<tr>
<td>Jupiter</td>
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<td>Saturn</td>
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<tr>
<td>Uranus</td>
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<tr>
<td>Neptune</td>
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<tr>
<td>Pluto</td>
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<tr>
<td>Halley’s comet</td>
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</tbody>
</table>

**POLAR EQUATIONS OF CONICS**

Our next objective is to derive polar equations for the conic sections from their focus–directrix characterizations. We will assume that the focus is at the pole and the directrix is either parallel or perpendicular to the polar axis. If the directrix is parallel to the polar axis, then it can be above or below the pole; and if the directrix is perpendicular to the polar axis, then it can be to the left or right of the pole. Thus, there are four cases to consider. We will derive the formulas for the case in which the directrix is perpendicular to the polar axis and to the right of the pole.

As illustrated in Figure 10.6.3, let us assume that the directrix is perpendicular to the polar axis and $d$ units to the right of the pole, where the constant $d$ is known. If $P$ is a point...
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on the conic and if the eccentricity of the conic is $e$, then it follows from Theorem 10.6.1 that $PF/ PD = e$ or, equivalently, that

$$PF = ePD$$  \hspace{1cm} (2)

However, it is evident from Figure 10.6.3 that $PF = r$ and $PD = d - r \cos \theta$. Thus, (2) can be written as

$$r = e(d - r \cos \theta)$$

which can be solved for $r$ and expressed as

$$r = \frac{ed}{1 + e \cos \theta}$$

(verify). Observe that this single polar equation can represent a parabola, an ellipse, or a hyperbola, depending on the value of $e$. In contrast, the rectangular equations for these conics all have different forms. The derivations in the other three cases are similar.

10.6.2 Theorem  If a conic section with eccentricity $e$ is positioned in a polar coordinate system so that its focus is at the pole and the corresponding directrix is $d$ units from the pole and is either parallel or perpendicular to the polar axis, then the equation of the conic has one of four possible forms, depending on its orientation:

<table>
<thead>
<tr>
<th>Focus</th>
<th>Directrix</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$ed$</td>
<td>$r = \frac{ed}{1 + e \cos \theta}$</td>
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<td>$r = \frac{ed}{1 - e \cos \theta}$</td>
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<td>$ed$</td>
<td>$r = \frac{ed}{1 + e \sin \theta}$</td>
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<td></td>
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<td>$r = \frac{ed}{1 - e \sin \theta}$</td>
</tr>
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**SKETCHING CONICS IN POLAR COORDINATES**

Precise graphs of conic sections in polar coordinates can be generated with graphing utilities. However, it is often useful to be able to make quick sketches of these graphs that show their orientations and give some sense of their dimensions. The orientation of a conic relative to the polar axis can be deduced by matching its equation with one of the four forms in Theorem 10.6.2. The key dimensions of a parabola are determined by the constant $p$ (Figure 10.4.5) and those of ellipses and hyperbolas by the constants $a, b,$ and $c$ (Figures 10.4.11 and 10.4.20). Thus, we need to show how these constants can be obtained from the polar equations.
10.6 Conic Sections in Polar Coordinates

**Example 1** Sketch the graph of \( r = \frac{2}{1 - \cos \theta} \) in polar coordinates.

**Solution.** The equation is an exact match to (4) with \( d = 2 \) and \( e = 1 \). Thus, the graph is a parabola with the focus at the pole and the directrix 2 units to the left of the pole. This tells us that the parabola opens to the right along the polar axis and \( p = 1 \). Thus, the parabola looks roughly like that sketched in Figure 10.6.4.

All of the important geometric information about an ellipse can be obtained from the values of \( a, b, \) and \( c \) in Figure 10.6.5. One way to find these values from the polar equation of an ellipse is based on finding the distances from the focus to the vertices. As shown in the figure, let \( r_0 \) be the distance from the focus to the closest vertex and \( r_1 \) the distance to the farthest vertex. Thus,

\[
r_0 = a - c \quad \text{and} \quad r_1 = a + c
\]

from which it follows that

\[
a = \frac{1}{2}(r_1 + r_0) \quad c = \frac{1}{2}(r_1 - r_0)
\]

Moreover, it also follows from (7) that

\[
r_0 r_1 = a^2 - c^2 = b^2
\]

Thus,

\[
b = \sqrt{r_0 r_1}
\]

**Example 2** Find the constants \( a, b, \) and \( c \) for the ellipse \( r = \frac{6}{2 + \cos \theta} \).

**Solution.** This equation does not match any of the forms in Theorem 10.6.2 because they all require a constant term of 1 in the denominator. However, we can put the equation into one of these forms by dividing the numerator and denominator by 2 to obtain

\[
r = \frac{3}{1 + \frac{1}{2} \cos \theta}
\]

This is an exact match to (3) with \( d = 6 \) and \( e = \frac{1}{2} \), so the graph is an ellipse with the directrix 6 units to the right of the pole. The distance \( r_0 \) from the focus to the closest vertex can be obtained by setting \( \theta = 0 \) in this equation, and the distance \( r_1 \) to the farthest vertex can be obtained by setting \( \theta = \pi \). This yields

\[
r_0 = \frac{3}{1 + \frac{1}{2} \cos 0} = \frac{3}{2} = 2, \quad r_1 = \frac{3}{1 + \frac{1}{2} \cos \pi} = \frac{3}{2} = 6
\]

Thus, from Formulas (8), (10), and (9), respectively, we obtain

\[
a = \frac{1}{2}(r_1 + r_0) = 4, \quad b = \sqrt{r_0 r_1} = 2\sqrt{3}, \quad c = \frac{1}{2}(r_1 - r_0) = 2
\]

The ellipse looks roughly like that sketched in Figure 10.6.6. \( \blacktriangleright \)

All of the important information about a hyperbola can be obtained from the values of \( a, b, \) and \( c \) in Figure 10.6.7. As with the ellipse, one way to find these values from the polar equation of a hyperbola is based on finding the distances from the focus to the vertices. As
shown in the figure, let \( r_0 \) be the distance from the focus to the closest vertex and \( r_1 \) the distance to the farthest vertex. Thus,

\[
  r_0 = c - a \quad \text{and} \quad r_1 = c + a
\]

(11)

from which it follows that

\[
  a = \frac{1}{2}(r_1 - r_0) \quad \text{and} \quad c = \frac{1}{2}(r_1 + r_0)
\]

(12–13)

Moreover, it also follows from (11) that

\[
  r_0 r_1 = c^2 - a^2 = b^2
\]

(14)

\[\begin{align*}
  a &= \frac{1}{2}(r_1 - r_0) \\
  c &= \frac{1}{2}(r_1 + r_0) \\
  r_0 r_1 &= c^2 - a^2 = b^2 \\
  b &= \sqrt{r_0 r_1}
\end{align*}\]

**Example 3** Sketch the graph of \( r = \frac{2}{1 + 2 \sin \theta} \) in polar coordinates.

**Solution.** This equation is an exact match to (5) with \( d = 1 \) and \( e = 2 \). Thus, the graph is a hyperbola with its directrix 1 unit above the pole. However, it is not so straightforward to compute the values of \( r_0 \) and \( r_1 \), since hyperbolas in polar coordinates are generated in a strange way as \( \theta \) varies from 0 to \( 2\pi \). This can be seen from Figure 10.6.8a, which is the graph of the given equation in rectangular \( \theta r \)-coordinates. It follows from this graph that the corresponding polar graph is generated in pieces (see Figure 10.6.8b):

- As \( \theta \) varies over the interval \( 0 \leq \theta < 7\pi/6 \), the value of \( r \) is positive and varies from 2 down to 2/3 and then to \( +\infty \), which generates part of the lower branch.
- As \( \theta \) varies over the interval \( 7\pi/6 < \theta \leq 3\pi/2 \), the value of \( r \) is negative and varies from \( -\infty \) to \( -2 \), which generates the right part of the upper branch.
- As \( \theta \) varies over the interval \( 3\pi/2 \leq \theta < 11\pi/6 \), the value of \( r \) is negative and varies from \( -2 \) to \( -\infty \), which generates the left part of the upper branch.
- As \( \theta \) varies over the interval \( 11\pi/6 < \theta \leq 2\pi \), the value of \( r \) is positive and varies from \( +\infty \) to 2, which fills in the missing piece of the lower right branch.

To obtain a rough sketch of a hyperbola, it is generally sufficient to locate the center, the asymptotes, and the points where \( \theta = 0, \theta = \pi/2, \theta = \pi, \) and \( \theta = 3\pi/2 \).

It is now clear that we can obtain \( r_0 \) by setting \( \theta = \pi/2 \) and \( r_1 \) by setting \( \theta = 3\pi/2 \). Keeping in mind that \( r_0 \) and \( r_1 \) are positive, this yields

\[
  r_0 = \frac{2}{1 + 2 \sin(\pi/2)} = \frac{2}{3}, \quad r_1 = \frac{2}{1 + 2 \sin(3\pi/2)} = \frac{2}{-1} = 2
\]
10.6 Conic Sections in Polar Coordinates

Thus, from Formulas (12), (14), and (13), respectively, we obtain

\[ a = \frac{1}{2} (r_1 - r_0) = \frac{2}{3}, \quad b = \sqrt{r_0 r_1} = \frac{2\sqrt{3}}{3}, \quad c = \frac{1}{2} (r_1 + r_0) = \frac{4}{3} \]

Thus, the hyperbola looks roughly like that sketched in Figure 10.6.8c.

APPLICATIONS IN ASTRONOMY

In 1609 Johannes Kepler published a book known as *Astronomia Nova* (or sometimes *Commentaries on the Motions of Mars*) in which he succeeded in distilling thousands of years of observational astronomy into three beautiful laws of planetary motion (Figure 10.6.9).

10.6.3 Kepler’s Laws

- **First law (Law of Orbits).** Each planet moves in an elliptical orbit with the Sun at a focus.
- **Second law (Law of Areas).** The radial line from the center of the Sun to the center of a planet sweeps out equal areas in equal times.
- **Third law (Law of Periods).** The square of a planet’s period (the time it takes the planet to complete one orbit about the Sun) is proportional to the cube of the semimajor axis of its orbit.

Kepler’s laws, although stated for planetary motion around the Sun, apply to all orbiting celestial bodies that are subjected to a single central gravitational force—artificial satellites subjected only to the central force of Earth’s gravity and moons subjected only to the central gravitational force of a planet, for example. Later in the text we will derive Kepler’s laws from basic principles, but for now we will show how they can be used in basic astronomical computations.

In an elliptical orbit, the closest point to the focus is called the **perigee** and the farthest point the **apogee** (Figure 10.6.10). The distances from the focus to the perigee and apogee...
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are called the **perigee distance** and **apogee distance**, respectively. For orbits around the Sun, it is more common to use the terms **perihelion** and **aphelion**, rather than perigee and apogee, and to measure time in Earth years and distances in astronomical units (AU), where 1 AU is the semimajor axis $a$ of the Earth’s orbit (approximately $150 \times 10^6$ km or $92.9 \times 10^6$ mi). With this choice of units, the constant of proportionality in Kepler’s third law is 1, since $a = 1$ AU produces a period of $T = 1$ Earth year. In this case Kepler’s third law can be expressed as

$$T = a^{3/2}$$

(15)

Shapes of elliptical orbits are often specified by giving the eccentricity $e$ and the semimajor axis $a$, so it is useful to express the polar equations of an ellipse in terms of these constants. Figure 10.6.11, which can be obtained from the ellipse in Figure 10.6.1 and the relationship $c = ea$, implies that the distance $d$ between the focus and the directrix is

$$d = a(e - a) = a(a - e) = a\left(1 - e^2\right)$$

(16)

from which it follows that $ed = a(1 - e^2)$. Thus, depending on the orientation of the ellipse, the formulas in Theorem 10.6.2 can be expressed in terms of $a$ and $e$ as

$$r = a\left(1 - e^2\right)$$

$$r = a\left(1 - e^2\right)$$

(17–18)

Moreover, it is evident from Figure 10.6.11 that the distances from the focus to the closest and farthest vertices can be expressed in terms of $a$ and $e$ as

$$r_0 = a - ea = a(1 - e)$$

and

$$r_1 = a + ea = a(1 + e)$$

(19–20)

**Example 4** Halley’s comet (last seen in 1986) has an eccentricity of 0.97 and a semimajor axis of $a = 18.1$ AU.

(a) Find the equation of its orbit in the polar coordinate system shown in Figure 10.6.12.

(b) Find the period of its orbit.

(c) Find its perihelion and aphelion distances.

**Solution (a).** From (17), the polar equation of the orbit has the form

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

But $a(1 - e^2) = 18.1(1 - (0.97)^2) \approx 1.07$. Thus, the equation of the orbit is

$$r = \frac{1.07}{1 + 0.97 \cos \theta}$$

**Solution (b).** From (15), with $a = 18.1$, the period of the orbit is

$$T = (18.1)^{3/2} \approx 77 \text{ years}$$

**Solution (c).** Since the perihelion and aphelion distances are the distances to the closest and farthest vertices, respectively, it follows from (19) and (20) that

$$r_0 = a - ea = a(1 - e) = 18.1(1 - 0.97) \approx 0.543 \text{ AU}$$

$$r_1 = a + ea = a(1 + e) = 18.1(1 + 0.97) \approx 35.7 \text{ AU}$$
or since 1 AU $\approx 150 \times 10^6$ km, the perihelion and aphelion distances in kilometers are
\[
\begin{align*}
    r_0 &= 18.1(1 - 0.97)(150 \times 10^6) \approx 81,500,000 \text{ km} \\
    r_1 &= 18.1(1 + 0.97)(150 \times 10^6) \approx 5,350,000,000 \text{ km}
\end{align*}
\]

\[\text{Example 5}\] An Apollo lunar lander orbits the Moon in an elliptic orbit with eccentricity $e = 0.12$ and semimajor axis $a = 2015$ km. Assuming the Moon to be a sphere of radius 1740 km, find the minimum and maximum heights of the lander above the lunar surface (Figure 10.6.13).

**Solution.** If we let $r_0$ and $r_1$ denote the minimum and maximum distances from the center of the Moon, then the minimum and maximum distances from the surface of the Moon will be
\[
\begin{align*}
    d_{\text{min}} &= r_0 - 1740 \\
    d_{\text{max}} &= r_1 - 1740
\end{align*}
\]
or from Formulas (19) and (20)
\[
\begin{align*}
    d_{\text{min}} &= r_0 - 1740 = a(1 - e) - 1740 = 2015(0.88) - 1740 = 33.2 \text{ km} \\
    d_{\text{max}} &= r_1 - 1740 = a(1 + e) - 1740 = 2015(1.12) - 1740 = 516.8 \text{ km}
\end{align*}
\]

**EXERCISE SET 10.6**  
**Graphing Utility**

1–2 Find the eccentricity and the distance from the pole to the directrix, and sketch the graph in polar coordinates.       
\[\text{Figure 10.6.13} \quad \text{[Image: NASA]}\]

1. (a) $r = \frac{3}{2 - 2 \cos \theta}$ \hspace{2cm} (b) $r = \frac{3}{2 + \sin \theta}$       
2. (a) $r = \frac{4}{2 + 3 \cos \theta}$ \hspace{2cm} (b) $r = \frac{5}{3 + 3 \sin \theta}$

3–4 Use Formulas (3)–(6) to identify the type of conic and its orientation. Check your answer by generating the graph with a graphing utility.  
\[\text{QUICK CHECK EXERCISES 10.6} \quad \text{(See page 763 for answers.)}\]

1. In each part, name the conic section described.
   (a) The set of points whose distance to the point (2, 3) is half the distance to the line $x + y = 1$ is _______.
   (b) The set of points whose distance to the point (2, 3) is equal to the distance to the line $x + y = 1$ is _______.
   (c) The set of points whose distance to the point (2, 3) is twice the distance to the line $x + y = 1$ is _______.

2. In each part: (i) Identify the polar graph as a parabola, an ellipse, or a hyperbola; (ii) state whether the directrix is above, below, to the left, or to the right of the pole; and (iii) find the distance from the pole to the directrix.
   (a) $r = \frac{1}{4 + \cos \theta}$ \hspace{2cm} (b) $r = \frac{1}{1 - 4 \cos \theta}$       
(c) $r = \frac{1}{4 + 4 \sin \theta}$ \hspace{2cm} (d) $r = \frac{4}{1 - \sin \theta}$

3. If the distance from a vertex of an ellipse to the nearest focus is $r_0$, and if the distance from that vertex to the farthest focus is $r_1$, then the semimajor axis is $a = _______ $ and the semiminor axis is $b = _______ $.

4. If the distance from a vertex of a hyperbola to the nearest focus is $r_0$, and if the distance from that vertex to the farthest focus is $r_1$, then the semifocal axis is $a = _______ $ and the semiconjugate axis is $b = _______ $.

5–6 Find a polar equation for the conic that has its focus at the pole and satisfies the stated conditions. Points are in polar coordinates and directrices in rectangular coordinates for simplicity. (In some cases there may be more than one conic that satisfies the conditions.)       

5. (a) Ellipse; $e = \frac{1}{2}$; directrix $x = 2$.
   (b) Parabola; directrix $x = 1$.
   (c) Hyperbola; $e = \frac{1}{2}$; directrix $y = 3$.       
6. (a) $r = \frac{4}{2 - 3 \sin \theta}$ \hspace{2cm} (b) $r = \frac{12}{4 + \cos \theta}$
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6. (a) Ellipse; ends of major axis \((2, π/2)\) and \((6, 3π/2)\).
    (b) Parabola; vertex \((2, π)\).
    (c) Hyperbola; \(e = √2\); vertex \((2, 0)\).

7–8 Find the distances from the pole to the vertices, and then apply Formulas (8)–(10) to find the equation of the ellipse in rectangular coordinates.

7. (a) \(r = \frac{6}{2 + \sin θ}\)
    (b) \(r = \frac{1}{2 - \cos θ}\)

8. (a) \(r = \frac{6}{5 + 2 \cos θ}\)
    (b) \(r = \frac{1}{4 - 3 \sin θ}\)

9–10 Find the distances from the pole to the vertices, and then apply Formulas (12)–(14) to find the equation of the hyperbola in rectangular coordinates.

9. (a) \(r = \frac{3}{1 + 2 \sin θ}\)
    (b) \(r = \frac{5}{2 - 3 \cos θ}\)

10. (a) \(r = \frac{4}{1 - 2 \sin θ}\)
    (b) \(r = \frac{15}{2 + 8 \cos θ}\)

11–12 Find a polar equation for the ellipse that has its focus at the pole and satisfies the stated conditions.

11. (a) Directrix to the right of the pole; \(a = 8\); \(e = \frac{1}{2}\).
    (b) Directrix below the pole; \(a = 4\); \(e = \frac{3}{5}\).

12. (a) Directrix to the left of the pole; \(b = 4\); \(e = \frac{3}{5}\).
    (b) Directrix above the pole; \(c = 5\); \(e = \frac{1}{2}\).

13. Find the polar equation of an equilateral hyperbola with a focus at the pole and vertex \((5, 0)\).

14. Prove that a hyperbola is an equilateral hyperbola if and only if \(e = √2\).

15. (a) Show that the coordinates of the point \(P\) on the hyperbola in Figure 10.6.1 satisfy the equation \(\sqrt{(x - c)^2 + y^2} = \frac{c}{a}x - a\).
    (b) Use the result obtained in part (a) to show that \(PF/PD = c/a\).

16. (a) Show that the eccentricity of an ellipse can be expressed in terms of \(r_0\) and \(r_1\) as \(e = \frac{r_1 - r_0}{r_1 + r_0}\).
    (b) Show that \(\frac{r_1}{r_0} = \frac{1 + e}{1 - e}\).

17. (a) Show that the eccentricity of a hyperbola can be expressed in terms of \(r_0\) and \(r_1\) as \(e = \frac{r_1 + r_0}{r_1 - r_0}\).
    (b) Show that \(\frac{r_1}{r_0} = \frac{e + 1}{e - 1}\).

18. (a) Sketch the curves
    \[ r = \frac{1}{1 + \cos θ} \] and \[ r = \frac{1}{1 - \cos θ} \]
    (b) Find polar coordinates of the intersections of the curves in part (a).
    (c) Show that the curves are orthogonal, that is, their tangent lines are perpendicular at the points of intersection.

19–22 True-False Determine whether the statement is true or false. Explain your answer.

19. If an ellipse is not a circle, then the eccentricity of the ellipse is less than one.

20. A parabola has eccentricity greater than one.

21. If one ellipse has foci that are farther apart than those of a second ellipse, then the eccentricity of the first is greater than that of the second.

22. If \(d\) is a positive constant, then the conic section with polar equation
    \[ r = \frac{d}{1 + \cos θ} \]
    is a parabola.

23–28 Use the following values, where needed:

- radius of the Earth = 4000 mi = 6440 km
- 1 year (Earth year) = 365 days (Earth days)
- 1 AU = 92.9 × 10^6 mi = 150 × 10^6 km

23. The dwarf planet Pluto has eccentricity \(e = 0.249\) and semimajor axis \(a = 39.5\) AU.
    (a) Find the period \(T\) in years.
    (b) Find the perihelion and aphelion distances.
    (c) Choose a polar coordinate system with the center of the Sun at the pole, and find a polar equation of Pluto’s orbit in that coordinate system.
    (d) Make a sketch of the orbit with reasonably accurate proportions.

24. (a) Let \(a\) be the semimajor axis of a planet’s orbit around the Sun, and let \(T\) be its period. Show that if \(T\) is measured in days and \(a\) is measured in kilometers, then \(T = (365 \times 10^{-9})(a/150)^{3/2}\).
    (b) Use the result in part (a) to find the period of the planet Mercury in days, given that its semimajor axis is \(a = 57.95 \times 10^6\) km.
    (c) Choose a polar coordinate system with the Sun at the pole, and find an equation for the orbit of Mercury in that coordinate system given that the eccentricity of the orbit is \(e = 0.206\).
    (d) Use a graphing utility to generate the orbit of Mercury from the equation obtained in part (c).

25. The Hale–Bopp comet, discovered independently on July 23, 1995 by Alan Hale and Thomas Bopp, has an orbital eccentricity of \(e = 0.9951\) and a period of 2380 years.
    (a) Find its semimajor axis in astronomical units (AU).
    (b) Find its perihelion and aphelion distances.
26. Mars has a perihelion distance of 204,520,000 km and an aphelion distance of 246,280,000 km.
(a) Use these data to calculate the eccentricity, and compare your answer to the value given in Table 10.6.1. 
(b) Find the period of Mars.
(c) Choose a polar coordinate system with the center of the Sun at the pole, and find an equation for the orbit of Mars in that coordinate system.
(d) Use a graphing utility to generate the orbit of Mars from the equation obtained in part (c).

27. **Vanguard 1** was launched in March 1958 into an orbit around the Earth with eccentricity $e = 0.21$ and semimajor axis 8864.5 km. Find the minimum and maximum heights of Vanguard 1 above the surface of the Earth.

28. The planet Jupiter is believed to have a rocky core of radius 10,000 km surrounded by two layers of hydrogen—a 40,000 km thick layer of compressed metallic-like hydrogen and a 20,000 km thick layer of ordinary molecular hydrogen. The visible features, such as the Great Red Spot, are at the outer surface of the molecular hydrogen layer.

### Chapter 10 Review Exercises

**Quick Check Answers 10.6**

1. (a) an ellipse  (b) a parabola  (c) a hyperbola
2. (a) (i) ellipse  (ii) to the right of the pole  (iii) distance = 3
   (b) (i) hyperbola  (ii) to the left of the pole  (iii) distance = 4
   (c) (i) parabola  (ii) above the pole  (iii) distance = 4
3. $\frac{1}{2}(r_1 + r_0)$; $\sqrt{ro^2}$
4. $\frac{1}{2}(r_1 - r_0)$; $\sqrt{ro^2}$

**CHAPTER 10 REVIEW EXERCISES**

1. Find parametric equations for the portion of the circle $x^2 + y^2 = 2$ that lies outside the first quadrant, oriented clockwise. Check your work by generating the curve with a graphing utility.

2. (a) Suppose that the equations $x = f(t)$, $y = g(t)$ describe a curve $C$ as $t$ increases from 0 to 1. Find parametric equations that describe the same curve $C$ but traced in the opposite direction as $t$ increases from 0 to 1.
   (b) Check your work using the parametric graphing feature of a graphing utility by generating the line segment between $(1, 2)$ and $(4, 0)$ in both possible directions as $t$ increases from 0 to 1.

3. (a) Find the slope of the tangent line to the parametric curve $x = t^3 + 1$, $y = t/2$ at $t = -1$ and $t = 1$ without eliminating the parameter.
   (b) Check your answers in part (a) by eliminating the parameter and differentiating a function of $x$.

4. Find $dy/dx$ and $d^2y/dx^2$ at $t = 2$ for the parametric curve $x = \frac{1}{2}t^2$, $y = \frac{1}{4}t^3$.

5. Find all values of $t$ at which a tangent line to the parametric curve $x = 2 \cos t$, $y = 4 \sin t$ is (a) horizontal  (b) vertical.

6. Find the exact arc length of the curve $x = 1 - 5t^4$, $y = 4t^3 - 1$ $(0 \leq t \leq 1)$

7. In each part, find the rectangular coordinates of the point whose polar coordinates are given.
   (a) $(8 \pi/4)$  (b) $(7, -\pi/4)$  (c) $(8, 9\pi/4)$
   (d) $(5, 0)$  (e) $(-2, -3\pi/2)$  (f) $(0, \pi)$

8. Express the point whose $xy$-coordinates are $(-1, 1)$ in polar coordinates with (a) $r > 0$, $0 \leq \theta < 2\pi$  (b) $r < 0$, $0 \leq \theta < 2\pi$
   (c) $r > 0$, $-\pi < \theta \leq \pi$  (d) $r < 0$, $-\pi < \theta \leq \pi$.

On November 6, 1997 the spacecraft **Galileo** was placed in a Jovian orbit to study the moon Europa. The orbit had eccentricity 0.814580 and semimajor axis 3,514,918.9 km. Find **Galileo’s** minimum and maximum heights above the molecular hydrogen layer (see the accompanying figure).
9. In each part, use a calculating utility to approximate the polar coordinates of the point whose rectangular coordinates are given.
(a) (4, 3)  
(b) (2, -5)  
(c) (1, \tan^{-1} 1)

10. In each part, state the name that describes the polar curve most precisely: a rose, a line, a circle, a limaçon, a cardioid, a spiral, a lemniscate, or none of these.
(a) \( r = 3 \cos \theta \)  
(b) \( r = \cos 3\theta \)  
(c) \( r = \cos \theta \)  
(d) \( r = 3 - \cos \theta \)  
(e) \( r = 1 - 3 \cos \theta \)  
(f) \( r^2 = 3 \cos \theta \)  
(g) \( r = (3 \cos \theta)^2 \)  
(h) \( r = 1 + 3\theta \)

11. In each part, identify the curve by converting the polar equation to rectangular coordinates. Assume that \( a > 0 \).
(a) \( r = a \sec^2 \frac{\theta}{2} \)  
(b) \( r^2 \cos 2\theta = a^2 \)  
(c) \( r = 4 \csc (\theta - \frac{\pi}{4}) \)  
(d) \( r = 4 \cos \theta + 8 \sin \theta \)

12. In each part, express the given equation in polar coordinates.
(a) \( x = 7 \)  
(b) \( x^2 + y^2 - 6y = 0 \)  
(c) \( x^2 + y^2 = 9 \)

13–17 Sketch the curve in polar coordinates.

13. \( \theta = \frac{\pi}{6} \)  
14. \( r = 6 \cos \theta \)  
15. \( r = 3(1 - \sin \theta) \)  
16. \( r^2 = \sin 2\theta \)  
17. \( r = 3 - \cos \theta \)

18. (a) Show that the maximum value of the \( y \)-coordinate of points on the curve \( r = 1/\sqrt{\theta} \) for \( \theta \) in the interval \( (0, \pi] \) occurs when \( \tan \theta = 2\theta \).
(b) Use a calculating utility to solve the equation in part (a) to at least four decimal-place accuracy.
(c) Use the result of part (b) to approximate the maximum value of \( y \) for \( 0 < \theta \leq \pi \).

19. (a) Find the minimum and maximum \( x \)-coordinates of points on the cardioid \( r = 1 - \cos \theta \).
(b) Find the minimum and maximum \( y \)-coordinates of points on the cardioid in part (a).

20. Determine the slope of the tangent line to the polar curve \( r = 1 + \sin \theta \) at \( \theta = \pi/4 \).

21. A parametric curve of the form

\[
x = a \cot t + b \cos t, \quad y = a + b \sin t \quad (0 < t < 2\pi)
\]

is called a conchoid of Nicomedes (see the accompanying figure for the case \( 0 < a < b \)).
(a) Describe how the conchoid

\[
x = \cot t + 4 \cos t, \quad y = 1 + 4 \sin t
\]

is generated as \( t \) varies over the interval \( 0 < t < 2\pi \).
(b) Find the horizontal asymptote of the conchoid given in part (a).
(c) For what values of \( t \) does the conchoid in part (a) have a horizontal tangent line? A vertical tangent line?

(d) Find a polar equation \( r = f(\theta) \) for the conchoid in part (a), and then find polar equations for the tangent lines to the conchoid at the pole.

![Figure Ex-21](image)

22. (a) Find the arc length of the polar curve \( r = 1/\theta \) for \( \pi/4 \leq \theta \leq \pi/2 \).
(b) What can you say about the arc length of the portion of the curve that lies inside the circle \( r = 1 \)?

23. Find the area of the region that is enclosed by the cardioid \( r = 2 + 2 \cos \theta \).

24. Find the area of the region in the first quadrant within the cardioid \( r = 1 + \sin \theta \).

25. Find the area of the region that is common to the circles \( r = 1, r = 2 \cos \theta, \) and \( r = 2 \sin \theta \).

26. Find the area of the region that is inside the cardioid \( r = a(1 + \sin \theta) \) and outside the circle \( r = a \sin \theta \).

27–30 Sketch the parabola, and label the focus, vertex, and directrix.

27. \( y^2 = 6x \)  
28. \( x^2 = -9y \)  
29. \( (y + 1)^2 = -7(x - 4) \)  
30. \( (x - 1)^2 = 2(y - 1) \)

31–34 Sketch the ellipse, and label the foci, the vertices, and the ends of the minor axis.

31. \( \frac{x^2}{4} + \frac{y^2}{25} = 1 \)  
32. \( 4x^2 + 9y^2 = 36 \)  
33. \( 9(x - 1)^2 + 16(y - 3)^2 = 144 \)  
34. \( 3(x + 2)^2 + 4(y + 1)^2 = 12 \)

35–37 Sketch the hyperbola, and label the vertices, foci, and asymptotes.

35. \( \frac{x^2}{16} - \frac{y^2}{4} = 1 \)  
36. \( 9y^2 - 4x^2 = 36 \)  
37. \( \frac{(x - 2)^2}{9} - \frac{(y - 4)^2}{4} = 1 \)

38. In each part, sketch the graph of the conic section with reasonably accurate proportions.
(a) \( x^2 - 4x + 8y + 36 = 0 \)  
(b) \( 3x^2 + 4y^2 - 30x - 8y + 67 = 0 \)  
(c) \( 4x^2 - 5y^2 - 8x - 30y - 21 = 0 \)

39–41 Find an equation for the conic described.

39. A parabola with vertex \((0, 0)\) and focus \((0, -4)\).
40. An ellipse with the ends of the major axis \((0, \pm \sqrt{3})\) and the ends of the minor axis \((\pm 1, 0)\).
41. A hyperbola with vertices \((0, \pm 3)\) and asymptotes \(y = \pm x\).
42. It can be shown in the accompanying figure that hanging cables form parabolic arcs rather than catenaries if they are subjected to uniformly distributed downward forces along their length. For example, if the weight of the roadway in a suspension bridge is assumed to be uniformly distributed along the supporting cables, then the cables can be modeled by parabolas.

(a) Assuming a parabolic model, find an equation for the cable in the accompanying figure, taking the y-axis to be vertical and the origin at the low point of the cable.

(b) Find the length of the cable between the supports.

43. It will be shown later in this text that if a projectile is launched with speed \( v_0 \) at an angle \( \alpha \) with the horizontal and at a height \( y_0 \) above ground level, then the resulting trajectory relative to the coordinate system in the accompanying figure will have parametric equations

\[
x = (v_0 \cos \alpha)t, \quad y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2
\]

where \( g \) is the acceleration due to gravity.

(a) Show that the trajectory is a parabola.

(b) Find the coordinates of the vertex.

44. Mickey Mantle is recognized as baseball's unofficial king of long home runs. On April 17, 1953 Mantle blasted a pitch by Chuck Stobbs of the hapless Washington Senators out of Griffith Stadium, just clearing the 50 ft wall at the 391 ft marker in left center. Assuming that the ball left the bat at a height of 3 ft above the ground and at an angle of 45°, use the parametric equations in Exercise 43 with \( g = 32 \text{ ft/s}^2 \) to find

(a) the speed of the ball as it left the bat

(b) the maximum height of the ball

(c) the distance along the ground from home plate to where the ball struck the ground.

45–47 Rotate the coordinate axes to remove the \( xy \)-term, and then name the conic.

45. \( x^2 + y^2 - 3xy - 3 = 0 \)
1. Recall from Section 5.10 that the Fresnel sine and cosine functions are defined as

\[ S(x) = \int_0^x \sin \left( \frac{\pi t^2}{2} \right) dt \quad \text{and} \quad C(x) = \int_0^x \cos \left( \frac{\pi t^2}{2} \right) dt \]

The following parametric curve, which is used to study amplitudes of light waves in optics, is called a clothoid or Cornu spiral in honor of the French scientist Marie Alfred Cornu (1841–1902):

\[ x = C(t) = \int_0^t \cos \left( \frac{\pi u^2}{2} \right) du \quad (-\infty < t < +\infty) \]

\[ y = S(t) = \int_0^t \sin \left( \frac{\pi u^2}{2} \right) du \]

(a) Use a CAS to graph the Cornu spiral.
(b) Describe the behavior of the spiral as \( t \to +\infty \) and as \( t \to -\infty \).
(c) Find the arc length of the spiral for \(-1 \leq t \leq 1\).

2. (a) The accompanying figure shows an ellipse with semimajor axis \( a \) and semiminor axis \( b \). Express the coordinates of the points \( P, Q, \) and \( R \) in terms of \( t \).
(b) How does the geometric interpretation of the parameter \( t \) differ between a circle

\[ x = a \cos t, \quad y = a \sin t \]

and an ellipse

\[ x = a \cos t, \quad y = b \sin t ? \]

3. The accompanying figure shows Kepler’s method for constructing a parabola. A piece of string the length of the left edge of the drafting triangle is tacked to the vertex \( Q \) of the triangle and the other end to a fixed point \( F \). A pencil holds the string taut against the base of the triangle as the edge opposite \( Q \) slides along a horizontal line \( L \) below \( F \). Show that the pencil traces an arc of a parabola with focus \( F \) and directrix \( L \).

4. The accompanying figure shows a method for constructing a hyperbola. A corner of a ruler is pinned to a fixed point \( F_1 \) and the ruler is free to rotate about that point. A piece of string whose length is less than that of the ruler is tacked to a point \( F_2 \) and to the free corner \( Q \) of the ruler on the same edge as \( F_1 \). A pencil holds the string taut against the top edge of the ruler as the ruler rotates about the point \( F_1 \). Show that the pencil traces an arc of a hyperbola with foci \( F_1 \) and \( F_2 \).

5. Consider an ellipse \( E \) with semimajor axis \( a \) and semiminor axis \( b \), and set \( c = \sqrt{a^2 - b^2} \).
(a) Show that the ellipsoid that results when \( E \) is revolved about its major axis has volume \( V = \frac{2}{3}\pi ab^2 \) and surface area

\[ S = 2\pi ab \left( \frac{b}{a} + \frac{a}{c} \sin^{-1} \frac{c}{a} \right) \]

(b) Show that the ellipsoid that results when \( E \) is revolved about its minor axis has volume \( V = \frac{2}{3}\pi a^2 b \) and surface area

\[ S = 2\pi ab \left( \frac{a}{b} + \frac{b}{c} \ln \frac{a + c}{b} \right) \]

EXPANDING THE CALCULUS HORIZON

To learn how polar coordinates and conic sections can be used to analyze the possibility of a collision between a comet and Earth, see the module entitled Comet Collision at:

www.wiley.com/college/anton