

SECTION P.5

Factoring Polynomials

Objectives

- 1 Factor out the greatest common factor of a polynomial.
- 2 Factor by grouping.
- 3 Factor trinomials.
- 4 Factor the difference of squares.
- 5 Factor perfect square trinomials.
- 6 Factor the sum or difference of two cubes.
- 7 Use a general strategy for factoring polynomials.
- 8 Factor algebraic expressions containing fractional and negative exponents.

A two-year-old boy is asked, “Do you have a brother?” He answers, “Yes.” “What is your brother’s name?” “Tom.” Asked if Tom has a brother, the two-year-old replies, “No.” The child can go in the direction from self to brother, but he cannot reverse this direction and move from brother back to self.

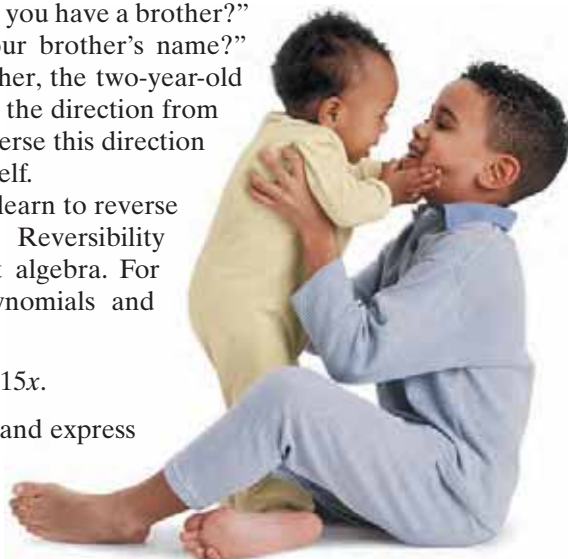
As our intellects develop, we learn to reverse the direction of our thinking. Reversibility of thought is found throughout algebra. For example, we can multiply polynomials and show that

$$5x(2x + 3) = 10x^2 + 15x.$$

We can also reverse this process and express the resulting polynomial as

$$10x^2 + 15x = 5x(2x + 3).$$

Factoring a polynomial expressed as the sum of monomials means finding an equivalent expression that is a product.

Factoring $10x^2 + 15x$

Sum of monomials

Equivalent expression that is a product

$$10x^2 + 15x = 5x(2x + 3)$$

The factors of $10x^2 + 15x$ are $5x$ and $2x + 3$.

In this section, we will be **factoring over the set of integers**, meaning that the coefficients in the factors are integers. Polynomials that cannot be factored using integer coefficients are called **irreducible over the integers**, or **prime**.

The goal in factoring a polynomial is to use one or more factoring techniques until each of the polynomial’s factors, except possibly for a monomial factor, is prime or irreducible. In this situation, the polynomial is said to be **factored completely**.

We will now discuss basic techniques for factoring polynomials.

- 1 Factor out the greatest common factor of a polynomial.

Common Factors

In any factoring problem, the first step is to look for the *greatest common factor*. The **greatest common factor**, abbreviated GCF, is an expression of the highest degree that divides each term of the polynomial. The distributive property in the reverse direction

$$ab + ac = a(b + c)$$

can be used to factor out the greatest common factor.

EXAMPLE 1 Factoring Out the Greatest Common Factor

Factor:

a. $18x^3 + 27x^2$

b. $x^2(x + 3) + 5(x + 3)$.

GREAT QUESTION!

Is there a rule that can help me determine the greatest common factor?

Yes. The variable part of the greatest common factor always contains the *smallest* power of a variable or algebraic expression that appears in all terms of the polynomial.

SOLUTION

a. First, determine the greatest common factor.

9 is the greatest integer that divides 18 and 27.

$$18x^3 + 27x^2$$

x^2 is the greatest expression that divides x^3 and x^2 .

The GCF of the two terms of the polynomial is $9x^2$.

$$\begin{aligned} 18x^3 + 27x^2 &= 9x^2(2x) + 9x^2(3) && \text{Express each term as the product} \\ & && \text{of the GCF and its other factor.} \\ &= 9x^2(2x + 3) && \text{Factor out the GCF.} \end{aligned}$$

b. In this situation, the greatest common factor is the common binomial factor $(x + 3)$. We factor out this common factor as follows:

$$x^2(x + 3) + 5(x + 3) = (x + 3)(x^2 + 5). \quad \text{Factor out the common binomial factor.}$$

 **Check Point 1** Factor:

a. $10x^3 - 4x^2$

b. $2x(x - 7) + 3(x - 7)$.

2 Factor by grouping.

Factoring by Grouping

Some polynomials have only a greatest common factor of 1. However, by a suitable grouping of the terms, it still may be possible to factor. This process, called **factoring by grouping**, is illustrated in Example 2.

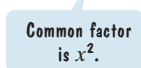
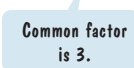
EXAMPLE 2 Factoring by Grouping

Factor: $x^3 + 4x^2 + 3x + 12$.

SOLUTION

There is no factor other than 1 common to all terms. However, we can group terms that have a common factor:

$$\boxed{x^3 + 4x^2} + \boxed{3x + 12}.$$

We now factor the given polynomial as follows:

$$\begin{aligned} x^3 + 4x^2 + 3x + 12 &= (x^3 + 4x^2) + (3x + 12) && \text{Group terms with common factors.} \\ &= x^2(x + 4) + 3(x + 4) && \text{Factor out the greatest common factor} \\ & && \text{from the grouped terms. The remaining two} \\ & && \text{terms have } x + 4 \text{ as a common binomial} \\ & && \text{factor.} \\ &= (x + 4)(x^2 + 3). && \text{Factor out the GCF, } x + 4. \end{aligned}$$

Thus, $x^3 + 4x^2 + 3x + 12 = (x + 4)(x^2 + 3)$. Check the factorization by multiplying the right side of the equation using the FOIL method. Because the factorization is correct, you should obtain the original polynomial.

DISCOVERY

In Example 2, group the terms as follows:

$$(x^3 + 3x) + (4x^2 + 12).$$

Factor out the greatest common factor from each group and complete the factoring process. Describe what happens. What can you conclude?

 **Check Point 2** Factor: $x^3 + 5x^2 - 2x - 10$.

 **Check Point 3** Factor: $x^2 + 13x + 40$.

EXAMPLE 4 Factoring a Trinomial Whose Leading Coefficient Is 1

Factor: $x^2 + 3x - 18$.

SOLUTION

Step 1 Find two First terms whose product is x^2 .

$$x^2 + 3x - 18 = (x \quad)(x \quad)$$

To find the second term of each factor, we must find two integers whose product is -18 and whose sum is 3.

Step 2 Find two Last terms whose product is -18 .

Factors of -18	18, -1	-18 , 1	9, -2	-9 , 2	6, -3	-6 , 3
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Step 3 Try various combinations of these factors. We are looking for the pair of factors whose sum is 3.

Factors of -18	18, -1	-18 , 1	9, -2	-9 , 2	6, -3	-6 , 3
Sum of Factors	17	-17	7	-7	3	-3

This is the desired sum.

Thus, $x^2 + 3x - 18 = (x + 6)(x - 3)$ or $(x - 3)(x + 6)$

GREAT QUESTION!

Is there a way to eliminate some of the combinations of factors for a trinomial whose leading coefficient is 1?

Yes. To factor $x^2 + bx + c$ when c is positive, find two numbers with the same sign as the middle term.

$$x^2 + 6x + 8 = (x + 4)(x + 2)$$

Same signs

$$x^2 - 5x + 6 = (x - 3)(x - 2)$$

Same signs

To factor $x^2 + bx + c$ when c is negative, find two numbers with opposite signs whose sum is the coefficient of the middle term.

$$x^2 + 3x - 18 = (x + 6)(x - 3)$$

Negative

Opposite signs

$$x^2 - 2x - 99 = (x + 9)(x - 11)$$

Negative

Opposite signs

 **Check Point 4** Factor: $x^2 - 5x - 14$.

EXAMPLE 5 Factoring a Trinomial Whose Leading Coefficient Is Not 1

Factor: $8x^2 - 10x - 3$.

GREAT QUESTION!

When factoring trinomials, must I list every possible factorization before getting the correct one?

With practice, you will find that it is not necessary to list every possible factorization of the trinomial. As you practice factoring, you will be able to narrow down the list of possible factors to just a few. When it comes to factoring, practice makes perfect.

SOLUTION

Step 1 Find two First terms whose product is $8x^2$.

$$8x^2 - 10x - 3 \stackrel{?}{=} (8x \quad)(x \quad)$$

$$8x^2 - 10x - 3 \stackrel{?}{=} (4x \quad)(2x \quad)$$

Step 2 Find two Last terms whose product is -3 . The possible factorizations are $1(-3)$ and $-1(3)$.

Step 3 Try various combinations of these factors. The correct factorization of $8x^2 - 10x - 3$ is the one in which the sum of the **Outside** and **Inside** products is equal to $-10x$. Here is a list of the possible factorizations:

Possible Factorizations of $8x^2 - 10x - 3$	Sum of Outside and Inside Products (Should Equal $-10x$)
$(8x + 1)(x - 3)$	$-24x + x = -23x$
$(8x - 3)(x + 1)$	$8x - 3x = 5x$
$(8x - 1)(x + 3)$	$24x - x = 23x$
$(8x + 3)(x - 1)$	$-8x + 3x = -5x$
$(4x + 1)(2x - 3)$	$-12x + 2x = -10x$
$(4x - 3)(2x + 1)$	$4x - 6x = -2x$
$(4x - 1)(2x + 3)$	$12x - 2x = 10x$
$(4x + 3)(2x - 1)$	$-4x + 6x = 2x$

These four factorizations use $(8x \quad)(x \quad)$ with $1(-3)$ and $-1(3)$ as factorizations of -3 .

These four factorizations use $(4x \quad)(2x \quad)$ with $1(-3)$ and $-1(3)$ as factorizations of -3 .

This is the required middle term.

Thus, $8x^2 - 10x - 3 = (4x + 1)(2x - 3)$ or $(2x - 3)(4x + 1)$.

Use FOIL multiplication to check either of these factorizations. ...

Check Point 5 Factor: $6x^2 + 19x - 7$.

GREAT QUESTION!

I zone out reading your long lists of possible factorizations. Are there any rules for shortening these lists?

Here are some suggestions for reducing the list of possible factorizations for $ax^2 + bx + c$:

1. If b is relatively small, avoid the larger factors of a .
2. If c is positive, the signs in both binomial factors must match the sign of b .
3. If the trinomial has no common factor, no binomial factor can have a common factor.
4. Reversing the signs in the binomial factors changes the sign of bx , the middle term.

EXAMPLE 6 Factoring a Trinomial in Two Variables

Factor: $2x^2 - 7xy + 3y^2$.

SOLUTION

Step 1 Find two First terms whose product is $2x^2$.

$$2x^2 - 7xy + 3y^2 = (2x \quad)(x \quad)$$

Step 2 Find two Last terms whose product is $3y^2$. The possible factorizations are $(y)(3y)$ and $(-y)(-3y)$.

Step 3 Try various combinations of these factors. The correct factorization of $2x^2 - 7xy + 3y^2$ is the one in which the sum of the **Outside** and **Inside** products is equal to $-7xy$. Here is a list of possible factorizations:

Possible Factorizations of $2x^2 - 7xy + 3y^2$	Sum of Outside and Inside Products (Should Equal $-7xy$)
$(2x + 3y)(x + y)$	$2xy + 3xy = 5xy$
$(2x + y)(x + 3y)$	$6xy + xy = 7xy$
$(2x - 3y)(x - y)$	$-2xy - 3xy = -5xy$
$(2x - y)(x - 3y)$	$-6xy - xy = -7xy$

This is the required middle term.

Thus,

$$2x^2 - 7xy + 3y^2 = (2x - y)(x - 3y) \quad \text{or} \quad (x - 3y)(2x - y).$$

Use FOIL multiplication to check either of these factorizations. ...

 **Check Point 6** Factor: $3x^2 - 13xy + 4y^2$.

4 Factor the difference of squares.

Factoring the Difference of Two Squares

A method for factoring the difference of two squares is obtained by reversing the special product for the sum and difference of two terms.

The Difference of Two Squares

If A and B are real numbers, variables, or algebraic expressions, then

$$A^2 - B^2 = (A + B)(A - B).$$

In words: The difference of the squares of two terms factors as the product of a sum and a difference of those terms.

EXAMPLE 7 Factoring the Difference of Two Squares

Factor: **a.** $x^2 - 4$ **b.** $81x^2 - 49$.

SOLUTION

We must express each term as the square of some monomial. Then we use the formula for factoring $A^2 - B^2$.

$$\mathbf{a.} \quad x^2 - 4 = x^2 - 2^2 = (x + 2)(x - 2)$$

$$A^2 - B^2 = (A + B)(A - B)$$

$$\mathbf{b.} \quad 81x^2 - 49 = (9x)^2 - 7^2 = (9x + 7)(9x - 7)$$
...

 **Check Point 7** Factor:

a. $x^2 - 81$

b. $36x^2 - 25$.

We have seen that a polynomial is factored completely when it is written as the product of prime polynomials. To be sure that you have factored completely, check to see whether any factors with more than one term in the factored polynomial can be factored further. If so, continue factoring.

GREAT QUESTION!

Why isn't factoring $x^4 - 81$ as $(x^2 + 9)(x^2 - 9)$ a complete factorization?

The second factor, $x^2 - 9$, is itself a difference of two squares and can be factored.

EXAMPLE 8 A Repeated Factorization

Factor completely: $x^4 - 81$.

SOLUTION

$$\begin{aligned} x^4 - 81 &= (x^2)^2 - 9^2 \\ &= (x^2 + 9)(x^2 - 9) \\ &= (x^2 + 9)(x^2 - 3^2) \\ &= (x^2 + 9)(x + 3)(x - 3) \end{aligned}$$

Express as the difference of two squares.

The factors are the sum and the difference of the expressions being squared.

The factor $x^2 - 9$ is the difference of two squares and can be factored.

The factors of $x^2 - 9$ are the sum and the difference of the expressions being squared. ...

 **Check Point 8** Factor completely: $81x^4 - 16$.

5 Factor perfect square trinomials.

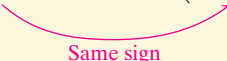
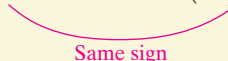
Factoring Perfect Square Trinomials

Our next factoring technique is obtained by reversing the special products for squaring binomials. The trinomials that are factored using this technique are called **perfect square trinomials**.

Factoring Perfect Square Trinomials

Let A and B be real numbers, variables, or algebraic expressions.

$$1. A^2 + 2AB + B^2 = (A + B)^2 \quad 2. A^2 - 2AB + B^2 = (A - B)^2$$

The two items in the box show that perfect square trinomials, $A^2 + 2AB + B^2$ and $A^2 - 2AB + B^2$, come in two forms: one in which the coefficient of the middle term is positive and one in which the coefficient of the middle term is negative. Here's how to recognize a perfect square trinomial:

1. The first and last terms are squares of monomials or integers.
2. The middle term is twice the product of the expressions being squared in the first and last terms.

EXAMPLE 9 Factoring Perfect Square Trinomials

Factor: **a.** $x^2 + 6x + 9$ **b.** $25x^2 - 60x + 36$.

SOLUTION

$$\mathbf{a.} \quad x^2 + 6x + 9 = x^2 + 2 \cdot x \cdot 3 + 3^2 = (x + 3)^2$$

The middle term has a positive sign.

$$A^2 + 2AB + B^2 = (A + B)^2$$

b. We suspect that $25x^2 - 60x + 36$ is a perfect square trinomial because $25x^2 = (5x)^2$ and $36 = 6^2$. The middle term can be expressed as twice the product of $5x$ and 6 .

$$25x^2 - 60x + 36 = (5x)^2 - 2 \cdot 5x \cdot 6 + 6^2 = (5x - 6)^2$$

$$A^2 - 2AB + B^2 = (A - B)^2$$

...

 **Check Point 9** Factor:

a. $x^2 + 14x + 49$

b. $16x^2 - 56x + 49$.

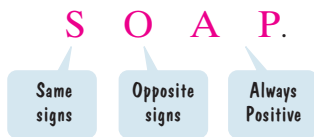
- 6 Factor the sum or difference of two cubes.

GREAT QUESTION!

A Cube of SOAP

The formulas for factoring $A^3 + B^3$ and $A^3 - B^3$ are difficult to remember and easy to confuse. Can you help me out?

When factoring sums or differences of cubes, observe the sign patterns shown by the voice balloons in the box. The word *SOAP* is a way to remember these patterns:



Factoring the Sum or Difference of Two Cubes

We can use the following formulas to factor the sum or the difference of two cubes:

Factoring the Sum or Difference of Two Cubes

1. Factoring the Sum of Two Cubes

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$



2. Factoring the Difference of Two Cubes

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$



EXAMPLE 10 Factoring Sums and Differences of Two Cubes

Factor: a. $x^3 + 8$ b. $64x^3 - 125$.

SOLUTION

- a. To factor $x^3 + 8$, we must express each term as the cube of some monomial. Then we use the formula for factoring $A^3 + B^3$.

$$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - x \cdot 2 + 2^2) = (x + 2)(x^2 - 2x + 4)$$

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

- b. To factor $64x^3 - 125$, we must express each term as the cube of some monomial. Then we use the formula for factoring $A^3 - B^3$.

$$64x^3 - 125 = (4x)^3 - 5^3 = (4x - 5)[(4x)^2 + (4x)(5) + 5^2]$$

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

$$= (4x - 5)(16x^2 + 20x + 25)$$

...

Check Point 10 Factor:

a. $x^3 + 1$

b. $125x^3 - 8$.

- 7 Use a general strategy for factoring polynomials.

A Strategy for Factoring Polynomials

It is important to practice factoring a wide variety of polynomials so that you can quickly select the appropriate technique. The polynomial is factored completely when all its polynomial factors, except possibly for monomial factors, are prime. Because of the commutative property, the order of the factors does not matter.

A Strategy for Factoring a Polynomial

- If there is a common factor, factor out the GCF.
- Determine the number of terms in the polynomial and try factoring as follows:
 - If there are two terms, can the binomial be factored by using one of the following special forms?

$$\text{Difference of two squares: } A^2 - B^2 = (A + B)(A - B)$$

$$\text{Sum of two cubes: } A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

$$\text{Difference of two cubes: } A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

- If there are three terms, is the trinomial a perfect square trinomial? If so, factor by using one of the following special forms:

$$A^2 + 2AB + B^2 = (A + B)^2$$

$$A^2 - 2AB + B^2 = (A - B)^2.$$

If the trinomial is not a perfect square trinomial, try factoring by trial and error.

- If there are four or more terms, try factoring by grouping.
- Check to see if any factors with more than one term in the factored polynomial can be factored further. If so, factor completely.

EXAMPLE 11 Factoring a Polynomial

Factor: $2x^3 + 8x^2 + 8x$.

SOLUTION

Step 1 If there is a common factor, factor out the GCF. Because $2x$ is common to all terms, we factor it out.

$$2x^3 + 8x^2 + 8x = 2x(x^2 + 4x + 4) \quad \text{Factor out the GCF.}$$

Step 2 Determine the number of terms and factor accordingly. The factor $x^2 + 4x + 4$ has three terms and is a perfect square trinomial. We factor using $A^2 + 2AB + B^2 = (A + B)^2$.

$$\begin{aligned} 2x^3 + 8x^2 + 8x &= 2x(x^2 + 4x + 4) \\ &= 2x(x^2 + 2 \cdot x \cdot 2 + 2^2) \\ &\quad \begin{array}{c} \text{A}^2 \quad + \quad 2AB \quad + \quad B^2 \end{array} \\ &= 2x(x + 2)^2 \end{aligned} \quad A^2 + 2AB + B^2 = (A + B)^2$$

Step 3 Check to see if factors can be factored further. In this problem, they cannot. Thus,

$$2x^3 + 8x^2 + 8x = 2x(x + 2)^2. \quad \dots$$

 **Check Point 11** Factor: $3x^3 - 30x^2 + 75x$.

EXAMPLE 12 Factoring a Polynomial

Factor: $x^2 - 25a^2 + 8x + 16$.

SOLUTION

Step 1 If there is a common factor, factor out the GCF. Other than 1 or -1 , there is no common factor.

Step 2 Determine the number of terms and factor accordingly. There are four terms. We try factoring by grouping. It can be shown that grouping into two groups of two terms does not result in a common binomial factor. Let's try grouping as a difference of squares.

$$\begin{aligned} x^2 - 25a^2 + 8x + 16 & \\ &= (x^2 + 8x + 16) - 25a^2 && \text{Rearrange terms and group as a perfect square trinomial minus } 25a^2 \text{ to obtain a difference of squares.} \\ &= (x + 4)^2 - (5a)^2 && \text{Factor the perfect square trinomial.} \\ &= (x + 4 + 5a)(x + 4 - 5a) && \text{Factor the difference of squares. The factors are the sum and difference of the expressions being squared.} \end{aligned}$$

Step 3 Check to see if factors can be factored further. In this case, they cannot, so we have factored completely. ...

 **Check Point 12** Factor: $x^2 - 36a^2 + 20x + 100$.

8 Factor algebraic expressions containing fractional and negative exponents.

Factoring Algebraic Expressions Containing Fractional and Negative Exponents

Although expressions containing fractional and negative exponents are not polynomials, they can be simplified using factoring techniques.

EXAMPLE 13 Factoring Involving Fractional and Negative Exponents

Factor and simplify: $x(x + 1)^{-\frac{3}{4}} + (x + 1)^{\frac{1}{4}}$.

SOLUTION

The greatest common factor of $x(x + 1)^{-\frac{3}{4}} + (x + 1)^{\frac{1}{4}}$ is $x + 1$ with the *smaller exponent* in the two terms. Thus, the greatest common factor is $(x + 1)^{-\frac{3}{4}}$.

$$\begin{aligned} x(x + 1)^{-\frac{3}{4}} + (x + 1)^{\frac{1}{4}} & \\ &= (x + 1)^{-\frac{3}{4}}x + (x + 1)^{-\frac{3}{4}}(x + 1)^{\frac{4}{4}} && \text{Express each term as the product of the greatest common factor and its other factor. Note that } (x + 1)^{-\frac{3}{4}}(x + 1)^{\frac{4}{4}} = (x + 1)^{-\frac{3}{4} + \frac{4}{4}} = (x + 1)^{\frac{1}{4}}. \\ &= (x + 1)^{-\frac{3}{4}}x + (x + 1)^{-\frac{3}{4}}(x + 1) && \text{Simplify: } (x + 1)^{\frac{4}{4}} = (x + 1). \\ &= (x + 1)^{-\frac{3}{4}}[x + (x + 1)] && \text{Factor out the greatest common factor.} \\ &= \frac{2x + 1}{(x + 1)^{\frac{3}{4}}} && b^{-n} = \frac{1}{b^n} \end{aligned}$$

 **Check Point 13** Factor and simplify: $x(x - 1)^{-\frac{1}{2}} + (x - 1)^{\frac{1}{2}}$

CONCEPT AND VOCABULARY CHECK

Here is a list of the factoring techniques that we have discussed.

- a. Factoring out the GCF
 b. Factoring by grouping
 c. Factoring trinomials by trial and error
 d. Factoring the difference of two squares

$$A^2 - B^2 = (A + B)(A - B)$$

- e. Factoring perfect square trinomials

$$A^2 + 2AB + B^2 = (A + B)^2$$

$$A^2 - 2AB + B^2 = (A - B)^2$$

- f. Factoring the sum of two cubes

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

- g. Factoring the difference of two cubes

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Fill in each blank by writing the letter of the technique (a through g) for factoring the polynomial.

1. $16x^2 - 25$ _____
2. $27x^3 - 1$ _____
3. $x^2 + 7x + xy + 7y$ _____
4. $4x^2 + 8x + 3$ _____
5. $9x^2 + 24x + 16$ _____
6. $5x^2 + 10x$ _____
7. $x^3 + 1000$ _____
8. The algebraic expression $(x + 1)^{\frac{1}{2}} - \frac{1}{3}(x + 1)^{\frac{3}{2}}$ can be factored using _____ as the greatest common factor.

EXERCISE SET P.5

Practice Exercises

In Exercises 1–10, factor out the greatest common factor.

- | | |
|-----------------------------|--------------------------------|
| 1. $18x + 27$ | 2. $16x - 24$ |
| 3. $3x^2 + 6x$ | 4. $4x^2 - 8x$ |
| 5. $9x^4 - 18x^3 + 27x^2$ | 6. $6x^4 - 18x^3 + 12x^2$ |
| 7. $x(x + 5) + 3(x + 5)$ | 8. $x(2x + 1) + 4(2x + 1)$ |
| 9. $x^2(x - 3) + 12(x - 3)$ | 10. $x^2(2x + 5) + 17(2x + 5)$ |

In Exercises 11–16, factor by grouping.

- | | |
|----------------------------|----------------------------|
| 11. $x^3 - 2x^2 + 5x - 10$ | 12. $x^3 - 3x^2 + 4x - 12$ |
| 13. $x^3 - x^2 + 2x - 2$ | 14. $x^3 + 6x^2 - 2x - 12$ |
| 15. $3x^3 - 2x^2 - 6x + 4$ | 16. $x^3 - x^2 - 5x + 5$ |

In Exercises 17–38, factor each trinomial, or state that the trinomial is prime.

- | | |
|-------------------------|-------------------------|
| 17. $x^2 + 5x + 6$ | 18. $x^2 + 8x + 15$ |
| 19. $x^2 - 2x - 15$ | 20. $x^2 - 4x - 5$ |
| 21. $x^2 - 8x + 15$ | 22. $x^2 - 14x + 45$ |
| 23. $3x^2 - x - 2$ | 24. $2x^2 + 5x - 3$ |
| 25. $3x^2 - 25x - 28$ | 26. $3x^2 - 2x - 5$ |
| 27. $6x^2 - 11x + 4$ | 28. $6x^2 - 17x + 12$ |
| 29. $4x^2 + 16x + 15$ | 30. $8x^2 + 33x + 4$ |
| 31. $9x^2 - 9x + 2$ | 32. $9x^2 + 5x - 4$ |
| 33. $20x^2 + 27x - 8$ | 34. $15x^2 - 19x + 6$ |
| 35. $2x^2 + 3xy + y^2$ | 36. $3x^2 + 4xy + y^2$ |
| 37. $6x^2 - 5xy - 6y^2$ | 38. $6x^2 - 7xy - 5y^2$ |

In Exercises 39–48, factor the difference of two squares.

- | | |
|--------------------|---------------------|
| 39. $x^2 - 100$ | 40. $x^2 - 144$ |
| 41. $36x^2 - 49$ | 42. $64x^2 - 81$ |
| 43. $9x^2 - 25y^2$ | 44. $36x^2 - 49y^2$ |

- | | |
|------------------|-----------------|
| 45. $x^4 - 16$ | 46. $x^4 - 1$ |
| 47. $16x^4 - 81$ | 48. $81x^4 - 1$ |

In Exercises 49–56, factor each perfect square trinomial.

- | | |
|----------------------|-----------------------|
| 49. $x^2 + 2x + 1$ | 50. $x^2 + 4x + 4$ |
| 51. $x^2 - 14x + 49$ | 52. $x^2 - 10x + 25$ |
| 53. $4x^2 + 4x + 1$ | 54. $25x^2 + 10x + 1$ |
| 55. $9x^2 - 6x + 1$ | 56. $64x^2 - 16x + 1$ |

In Exercises 57–64, factor using the formula for the sum or difference of two cubes.

- | | |
|------------------|------------------|
| 57. $x^3 + 27$ | 58. $x^3 + 64$ |
| 59. $x^3 - 64$ | 60. $x^3 - 27$ |
| 61. $8x^3 - 1$ | 62. $27x^3 - 1$ |
| 63. $64x^3 + 27$ | 64. $8x^3 + 125$ |

In Exercises 65–92, factor completely, or state that the polynomial is prime.

- | | |
|-----------------------------------|------------------------------|
| 65. $3x^3 - 3x$ | 66. $5x^3 - 45x$ |
| 67. $4x^2 - 4x - 24$ | 68. $6x^2 - 18x - 60$ |
| 69. $2x^4 - 162$ | 70. $7x^4 - 7$ |
| 71. $x^3 + 2x^2 - 9x - 18$ | 72. $x^3 + 3x^2 - 25x - 75$ |
| 73. $2x^2 - 2x - 112$ | 74. $6x^2 - 6x - 12$ |
| 75. $x^3 - 4x$ | 76. $9x^3 - 9x$ |
| 77. $x^2 + 64$ | 78. $x^2 + 36$ |
| 79. $x^3 + 2x^2 - 4x - 8$ | 80. $x^3 + 2x^2 - x - 2$ |
| 81. $y^5 - 81y$ | 82. $y^5 - 16y$ |
| 83. $20y^4 - 45y^2$ | 84. $48y^4 - 3y^2$ |
| 85. $x^2 - 12x + 36 - 49y^2$ | 86. $x^2 - 10x + 25 - 36y^2$ |
| 87. $9b^2x - 16y - 16x + 9b^2y$ | |
| 88. $16a^2x - 25y - 25x + 16a^2y$ | |

89. $x^2y - 16y + 32 - 2x^2$ 90. $12x^2y - 27y - 4x^2 + 9$
 91. $2x^3 - 8a^2x + 24x^2 + 72x$
 92. $2x^3 - 98a^2x + 28x^2 + 98x$

In Exercises 93–102, factor and simplify each algebraic expression.

93. $x^{\frac{3}{2}} - x^{\frac{1}{2}}$ 94. $x^{\frac{3}{4}} - x^{\frac{1}{4}}$
 95. $4x^{-\frac{2}{3}} + 8x^{\frac{1}{3}}$ 96. $12x^{-\frac{3}{4}} + 6x^{\frac{1}{4}}$
 97. $(x + 3)^{\frac{1}{2}} - (x + 3)^{\frac{3}{2}}$ 98. $(x^2 + 4)^{\frac{3}{2}} + (x^2 + 4)^{\frac{7}{2}}$
 99. $(x + 5)^{-\frac{1}{2}} - (x + 5)^{-\frac{3}{2}}$
 100. $(x^2 + 3)^{-\frac{2}{3}} + (x^2 + 3)^{-\frac{5}{3}}$
 101. $(4x - 1)^{\frac{1}{2}} - \frac{1}{3}(4x - 1)^{\frac{3}{2}}$
 102. $-8(4x + 3)^{-2} + 10(5x + 1)(4x + 3)^{-1}$

Practice Plus

In Exercises 103–114, factor completely.

103. $10x^2(x + 1) - 7x(x + 1) - 6(x + 1)$
 104. $12x^2(x - 1) - 4x(x - 1) - 5(x - 1)$
 105. $6x^4 + 35x^2 - 6$ 106. $7x^4 + 34x^2 - 5$
 107. $y^7 + y$ 108. $(y + 1)^3 + 1$
 109. $x^4 - 5x^2y^2 + 4y^4$ 110. $x^4 - 10x^2y^2 + 9y^4$
 111. $(x - y)^4 - 4(x - y)^2$ 112. $(x + y)^4 - 100(x + y)^2$
 113. $2x^2 - 7xy^2 + 3y^4$ 114. $3x^2 + 5xy^2 + 2y^4$

Application Exercises

115. Your computer store is having an incredible sale. The price on one model is reduced by 40%. Then the sale price is reduced by another 40%. If x is the computer's original price, the sale price can be modeled by

$$(x - 0.4x) - 0.4(x - 0.4x).$$

- a. Factor out $(x - 0.4x)$ from each term. Then simplify the resulting expression.
 b. Use the simplified expression from part (a) to answer these questions. With a 40% reduction followed by a 40% reduction, is the computer selling at 20% of its original price? If not, at what percentage of the original price is it selling?

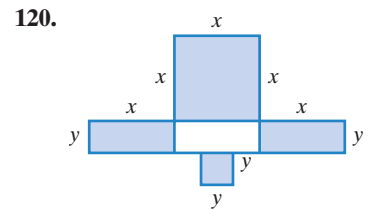
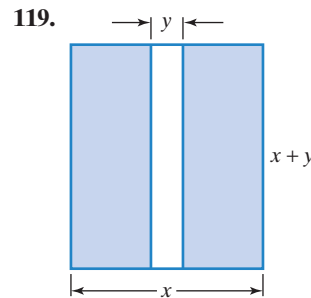
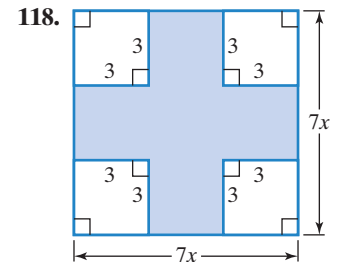
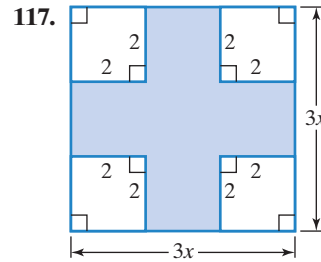
116. Your local electronics store is having an end-of-the-year sale. The price on a plasma television had been reduced by 30%. Now the sale price is reduced by another 30%. If x is the television's original price, the sale price can be modeled by

$$(x - 0.3x) - 0.3(x - 0.3x).$$

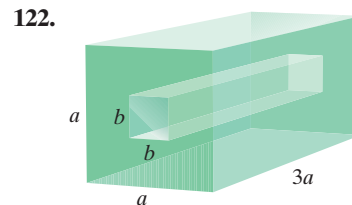
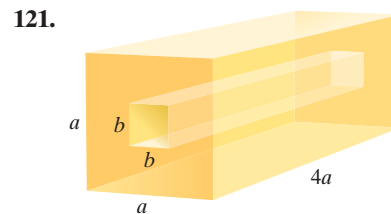
- a. Factor out $(x - 0.3x)$ from each term. Then simplify the resulting expression.
 b. Use the simplified expression from part (a) to answer these questions. With a 30% reduction followed by a 30% reduction, is the television selling at 40% of its original price? If not, at what percentage of the original price is it selling?

In Exercises 117–120,

- a. Write an expression for the area of the shaded region.
 b. Write the expression in factored form.



In Exercises 121–122, find the formula for the volume of the region outside the smaller rectangular solid and inside the larger rectangular solid. Then express the volume in factored form.



Writing in Mathematics

123. Using an example, explain how to factor out the greatest common factor of a polynomial.
 124. Suppose that a polynomial contains four terms. Explain how to use factoring by grouping to factor the polynomial.
 125. Explain how to factor $3x^2 + 10x + 8$.
 126. Explain how to factor the difference of two squares. Provide an example with your explanation.
 127. What is a perfect square trinomial and how is it factored?
 128. Explain how to factor $x^3 + 1$.
 129. What does it mean to factor completely?

32. $64y - y^4$
 33. $50x^3 + 20x^2 + 2x$
 34. $x^2 - 6x + 9 - 49y^2$

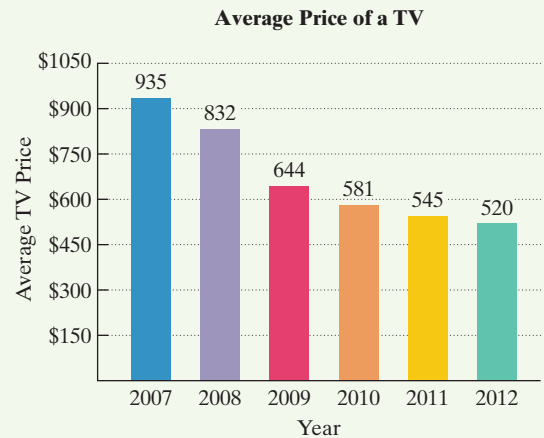
In Exercises 35–36, factor and simplify each algebraic expression.

35. $x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}} + x^{\frac{1}{2}}$
 36. $(x^2 + 1)^{\frac{1}{2}} - 10(x^2 + 1)^{-\frac{1}{2}}$
 37. List all the rational numbers in this set:

$$\left\{ -11, -\frac{3}{7}, 0, 0.45, \sqrt{23}, \sqrt{25} \right\}.$$

In Exercises 38–39, rewrite each expression without absolute value bars.

38. $|2 - \sqrt{13}|$
 39. $x^2|x|$ if $x < 0$
 40. If the population of the United States is approximately 3.0×10^8 and each person produces about 4.6 pounds of garbage per day, express the total number of pounds of garbage produced in the United States in one day in scientific notation.
 41. A human brain contains 3×10^{10} neurons and a gorilla brain contains 7.5×10^9 neurons. How many times as many neurons are in the brain of a human as in the brain of a gorilla?
 42. TVs keep getting fancier and bigger, but prices do not. The bar graph at the top of the next column shows the average price of a TV in the United States from 2007 through 2012.



Source: Consumer Electronics Association

Here are two mathematical models for the data shown by the graph. In each formula, P represents the average price of a TV x years after 2007.

Model 1 $P = -86x + 890$

Model 2 $P = 18x^2 - 175x + 950$

- a. Which model better describes the data for 2007?
 b. Does the polynomial model of degree 2 underestimate or overestimate the average TV price for 2012? By how much?

SECTION P.6

Rational Expressions

Objectives

- Specify numbers that must be excluded from the domain of a rational expression.
- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.
- Simplify fractional expressions that occur in calculus.
- Rationalize numerators.

How do we describe the costs of reducing environmental pollution? We often use algebraic expressions involving quotients of polynomials. For example, the algebraic expression

$$\frac{250x}{100 - x}$$

describes the cost, in millions of dollars, to remove x percent of the pollutants that are discharged into a river. Removing a modest percentage of pollutants, say 40%, is far less costly than removing a substantially greater percentage, such as 95%. We see this by evaluating the algebraic expression for $x = 40$ and $x = 95$.

Evaluating $\frac{250x}{100 - x}$ for

$x = 40$:

Cost is $\frac{250(40)}{100 - 40} \approx 167$.

$x = 95$:

Cost is $\frac{250(95)}{100 - 95} = 4750$.

The cost increases from approximately \$167 million to a possibly prohibitive \$4750 million, or \$4.75 billion. Costs spiral upward as the percentage of removed pollutants increases.

